New Ideas for Teaching Relativity:

Space-Time Trigonometry

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- **In the teaching of Relativity, there are many references to analogues:**
  - **physical concepts in Galilean and Special Relativity**
  - **mathematical concepts in Euclidean and Minkowskian geometry**

- **There is a known (but not well-known) relationship among the Euclidean, Galilean and Minkowskian geometries:**
  - **They are the "affine Cayley-Klein geometries".**
    (The de Sitter spacetimes are among the Cayley-Klein geometries.)

- **We exploit this fact to develop a new presentation of relativity, which may be useful for teaching high-school and college students.**
- **eventual Goal:**
  Introduce *some* **"spacetime intuition" earlier in the curriculum.**
- *Goal for this poster:* **Highlight the foundations of this approach.**
- **What makes this formulation interesting is that**
  - *the* **geometry of Galilean Relativity** *acts like a "bridge" from Euclidean geometry to Special Relativity.*
  - *a faithful* **visualization of tensor-algebra** *can be incorporated*

A trigonometric analogy (Yaglom)[a]

[a] I.M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis* (1979).

Euclidean rotation

$$t' = (\cos\theta)t + (-\sin\theta)y \qquad t' = \left(\frac{1}{\sqrt{1+v^2}}\right)t + \left(\frac{-v}{\sqrt{1+v^2}}\right)y$$

$$y' = (\sin\theta)t + (\ \ \cos\theta)y \qquad y' = \left(\frac{v}{\sqrt{1+v^2}}\right)t + \left(\frac{1}{\sqrt{1+v^2}}\right)y$$

where $v = \tan\theta$.

Galilean boost transformation

$$t' = (\textbf{cosg}\ \theta)t + (0\,\textbf{sing}\ \theta)y \qquad t' = \left(\ 1\ \right)t$$

$$y' = (\textbf{sing}\ \theta)t + (\ \textbf{cosg}\ \theta)y \qquad y' = \left(\ v\ \right)t + \left(\ 1\ \right)y$$

where $v = \textbf{tang}\ \theta$.

Yaglom defines **cosg** $\theta \equiv 1$, **sing** $\theta \equiv \theta$ so that **tang** $\theta \equiv \frac{\textbf{sing}\ \theta}{\textbf{cosg}\ \theta} \equiv \theta$.

Lorentz boost transformation

$$t' = (\cosh\theta)t + (+\sinh\theta)y \qquad t' = \left(\frac{1}{\sqrt{1-v^2}}\right)t + \left(\frac{+v}{\sqrt{1-v^2}}\right)y$$

$$y' = (\sinh\theta)t + (\ \ \cosh\theta)y \qquad y' = \left(\frac{v}{\sqrt{1-v^2}}\right)t + \left(\frac{1}{\sqrt{1-v^2}}\right)y$$

where $v = \tanh\theta$.

The Cayley-Klein Geometries

	measure of length between two points		
measure of angle between two lines	elliptic ($\eta^2 = -1$)	parabolic ($\eta^2 = 0$)	hyperbolic ($\eta^2 = +1$)
elliptic ($\epsilon^2 = -1$)	**Elliptic**	**Euclidean**	**Hyperbolic**
parabolic ($\epsilon^2 = 0$)	co-Euclidean "anti-Newton-Hooke"	**Galilean**	co-Minkowskian "Newton-Hooke"
hyperbolic ($\epsilon^2 = +1$)	anti-De-Sitter	**Minkowski**	De-Sitter

2-dimensional manifolds with

metric signature $(+1, -\epsilon^2)$ and constant curvature $\kappa = -\eta^2$

$$ds^2 = g_{ab}\, dx^a dx^b$$
$$= \frac{(\mathbf{1} + \eta^{\mathbf{2}}\epsilon^{\mathbf{2}}\mathbf{y}^{\mathbf{2}})\mathbf{dt}^{\mathbf{2}} - (\mathbf{1} - \eta^{\mathbf{2}}\mathbf{t}^{\mathbf{2}})\epsilon^{\mathbf{2}}\mathbf{dy}^{\mathbf{2}} - \mathbf{2}\eta^{\mathbf{2}}\epsilon^{\mathbf{2}}\mathbf{ty\ dt\ dy}}{(\mathbf{1} - \eta^{\mathbf{2}}(\mathbf{t}^{\mathbf{2}} - \epsilon^{\mathbf{2}}\mathbf{y}^{\mathbf{2}}))^{\mathbf{2}}}$$

Column $\eta^2 = 0$ are the affine Cayley-Klein geometries.

Row $\epsilon^2 = -1$ includes the classical non-Euclidean geometries.

Row $\epsilon^2 = +1$ are the constant curvature Lorentzian spacetimes.

The Metric
(Proper Time)

We will be concerned with the "$\eta^2 = 0$" (or "affine") geometries. [Observe that Euclid V, the Parallel Postulate, is valid for these geometries.] In this case, the line-element reduces to

$$\boxed{dS^2 = (dt)^2 - \epsilon^2 (dy)^2}$$

which is a unified (meta-)expression for

$$\left\{ \begin{array}{ll} (ds^2)_{Euc} = (dt)^2 + (dy)^2 & \text{if } \epsilon^2 = -1 \\ (ds^2)_{Gal} = (dt)^2 & \text{if } \epsilon^2 = 0 \\ (ds^2)_{Min} = (dt)^2 - (dy)^2 & \text{if } \epsilon^2 = +1 \end{array} \right. .$$

Upon introducing the metric tensor, the line-element can be written

$$dS^2 = \begin{pmatrix} dt \\ dy \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon^2 \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix}.$$

Since the Galilean metric is degenerate, one needs to provide an additional metric in order to measure spacelike separations:

$$dL^2 = \left\{ \begin{array}{ll} -\dfrac{1}{\epsilon^2} dS^2 & \text{if } \epsilon^2 \neq 0 \\ (dy)^2 & \text{if } \epsilon^2 = 0 \end{array} \right. .$$

The Circle

Define the "CIRCLE of radius R" to be the locus of points (t, y) that is a constant positive square-interval R^2 from a common point (t_0, y_0).

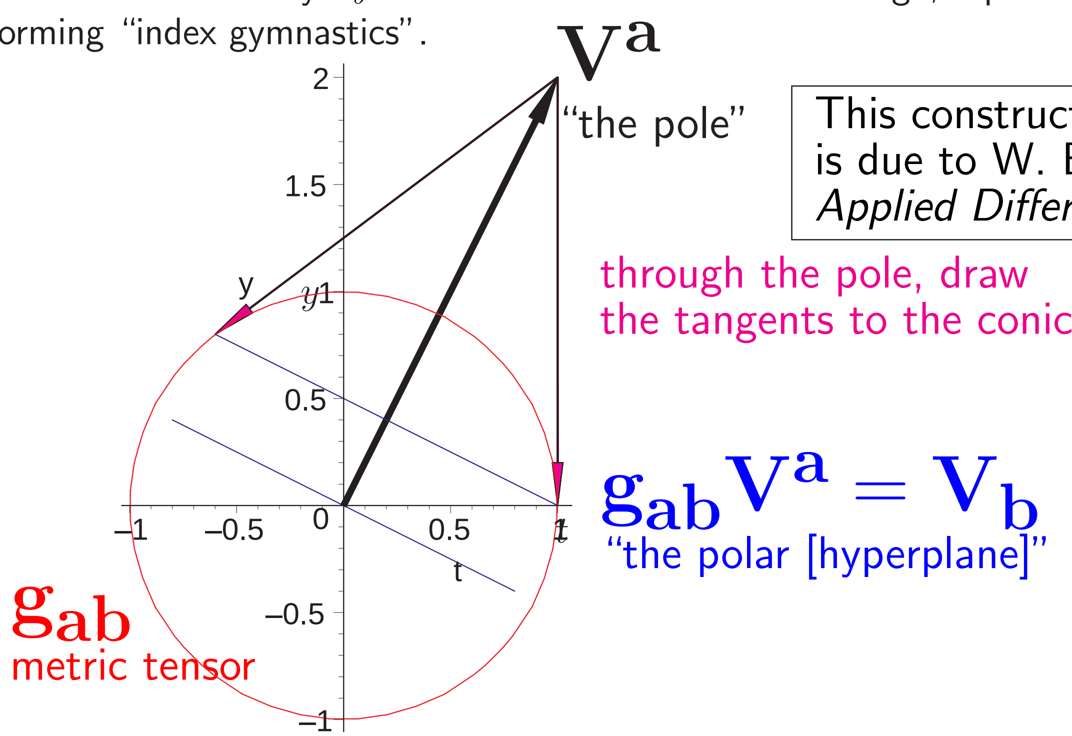
$$\boxed{R^2 = (t - t_0)^2 - \epsilon^2 (y - y_0)^2}$$

In fact, the unit-circle [generally, unit-sphere] provides a faithful visualization of a symmetric metric tensor.

The Pole and the Polar
(Visualizing Tensor-Index raising and lowering)

A metric tensor is a symmetric tensor that can be used to assign magnitudes to vectors.

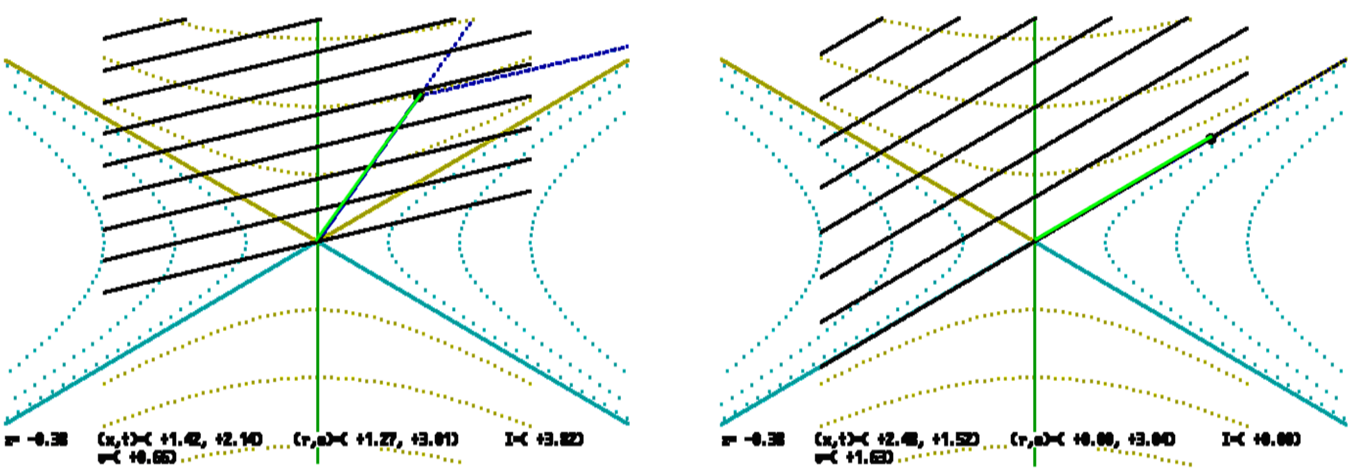
A metric tensor can also provide a **rule to identify a vector with a unique covector.** The vector and its covector are [metric-]duals of each other with this metric. Given a vector V^a, in the presence of a metric g_{ab}, we can form the combination $g_{ab}V^a$, which is a covector denoted by V_b. This is known as "index lowering", a particular move when performing "index gymnastics".



This construction
is due to W. Burke,
Applied Differential Geometry

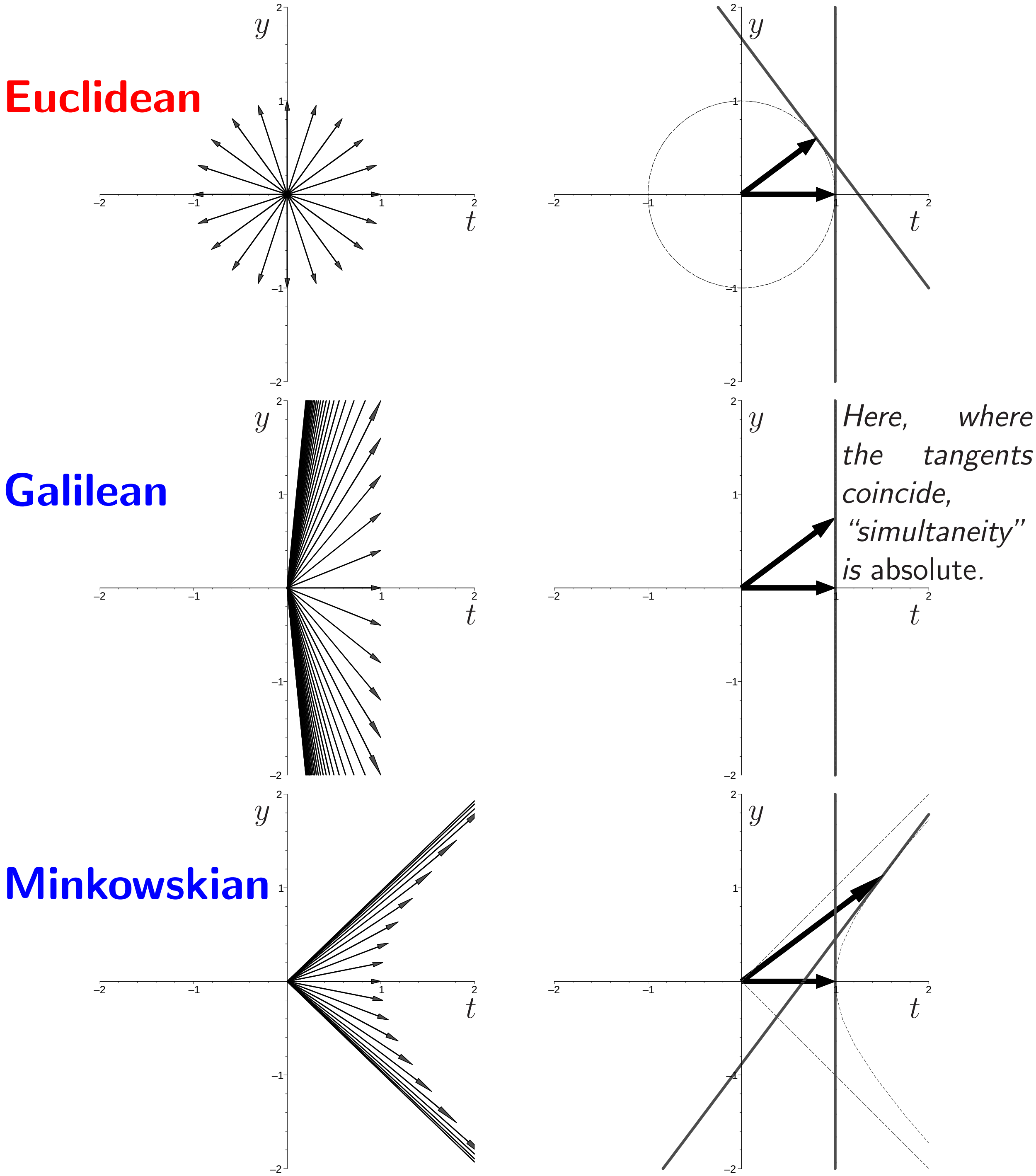
$$\left(\begin{array}{c}\text{the number of polar hyperplanes of } V_b \\ \text{pierced by the vector } V^a \\ \text{(number of "bongs of bell")}\end{array}\right) = \left(\begin{array}{c}\text{square-norm of the vector } V^a \\ g_{ab}V^aV^b\end{array}\right)$$

In Minkowski spacetime: a timelike vector and a lightlike vector and their metric-duals.



The Trilogy of the Surveyors

a draft of the Trilogy is available

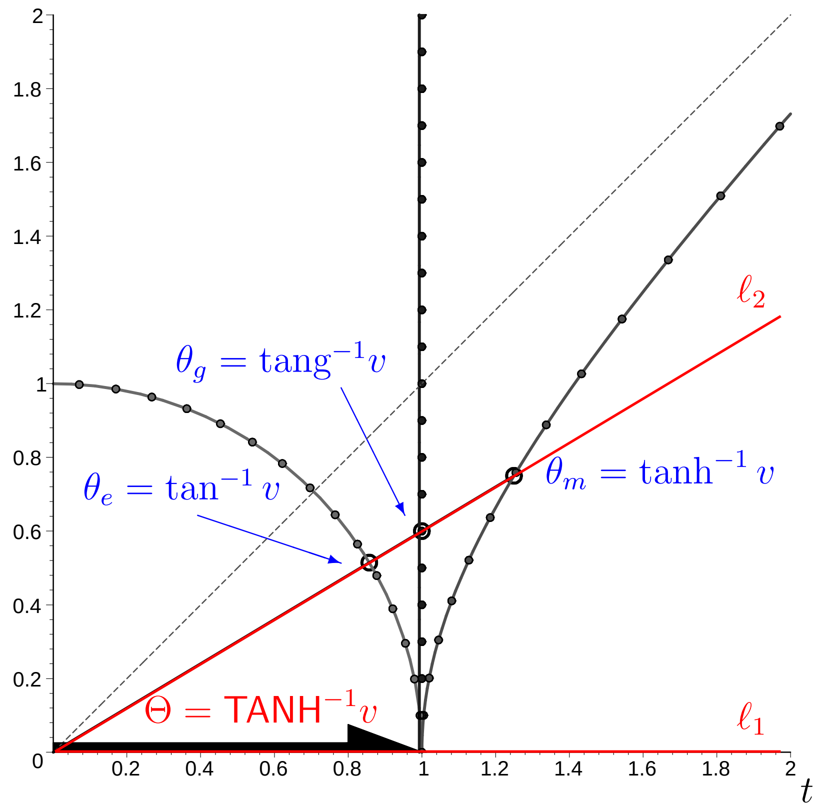


Family of **surveyors** of a two-dimensional plane:
From this **point**, travel in all possible **directions**. Stop when your **odometer** reads $1\ \mathrm{mi}$.
This "calibration curve" defines a **circle**, with **perpendicular** as "tangent to the circle".
Family of **observers** of a two-dimensional spacetime:
From this **event**, travel with all possible **velocities**. Stop when your **wristwatch** reads $1\ \mathrm{s}$.
This "calibration curve" defines a **"circle"**, with **simultaneous** as "tangent to the circle".

The Angle
(Rapidity)

Define the "ANGLE-measure between two future-timelike lines ℓ_1 and ℓ_2":

$$\boldsymbol{\Theta = \frac{\text{CIRCULAR arc-LENGTH } L \text{ intercepted by } \ell_1 \text{ and } \ell_2}{\text{radius } R \text{ of the CIRCLE centered at } o}}$$



Use the [spacelike] square-interval to measure the spacelike arc-length along the circle $t^2 - \epsilon^2 y^2 = R^2$:

$$\Theta = \frac{1}{R} \int dL$$

For $\epsilon^2 \neq 0$ cases,

$$\Theta = \frac{1}{R}\int\sqrt{dy^2 - \frac{1}{\epsilon^2}dt^2} = \int \frac{dy}{\sqrt{R^2+\epsilon^2 y^2}} = \frac{1}{\epsilon}\sinh^{-1}(\epsilon y/R)$$

Thus,

$$\begin{aligned} \epsilon y &= R\sinh(\epsilon\Theta) \\ t &= R\cosh(\epsilon\Theta). \end{aligned}$$

Euclidean case $(\epsilon^2 = -1)$

$$y = R\sin(\theta_e)$$
$$t = R\cos(\theta_e)$$

Minkowskian case $(\epsilon^2 = +1)$

$$y = R\sinh(\theta_m)$$
$$t = R\cosh(\theta_m)$$

For the Galilean case $(\epsilon^2 = 0)$,

$$\theta_g = \frac{1}{R}\int dL = \frac{1}{R}\int dy = \frac{y}{R}.$$

Thus,

$$\begin{aligned} y = R\theta_g &= R\ \mathbf{sing}(\theta_g) \\ t = R &= R\ \mathbf{cosg}(\theta_g) \end{aligned}$$

where $\mathbf{cosg}(\theta_g) = 1$ and $\mathbf{sing}(\theta_g) = \theta_g$

We can write the results for the three cases as

$$\boxed{\begin{aligned} y &= R\ \mathbf{SINH}\,\Theta \\ t &= R\ \mathbf{COSH}\,\Theta \end{aligned}}$$

the connecting relation ("velocity = tangent(rapidity)")

$$\mathbf{v} = \frac{\mathbf{y}}{\mathbf{t}} = \tan\theta_{\mathbf{e}} = \operatorname{tang}\theta_{\mathbf{g}} = \tanh\theta_{\mathbf{m}}$$

Hypercomplex Numbers?
(Maximum Signal Speed)

Physically, $\epsilon^2 = (c_{light}/c_{max})^2$

Minkowskian	$c_{max} = c_{light}$ (finite):	$\epsilon^2 = 1$ [but $\epsilon \neq 1$] (double numbers)
Galilean	$c_{max} = \infty$ (infinite):	$\epsilon^2 = 0$ [but $\epsilon \neq 0$] (dual numbers)
Euclidean	$c_{max} = ic_{light}$ (finite, imaginary):	$\epsilon^2 = -1$ (complex numbers)

It is convenient (but not necessary) to introduce the following "generalized complex" or "hypercomplex" number systems. Consider quantities of the form $\mathbf{z} = \mathbf{a} + \epsilon\mathbf{b}$, where a and b are real-numbers and ϵ is the "generalized imaginary number". These quantities can be given a matrix representation:

real	$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	
complex	$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	**Euclidean**
dual ($\epsilon^2 = 0$ [$\epsilon \neq 0$])	$\epsilon = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$	**Galilean**
double ($\epsilon^2 = 1$ [$\epsilon \neq 1$])	$\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$	**Minkowskian**

(these number systems have "divisors of zero")

Dual ($\epsilon^2 = 0$ [$\epsilon \neq 0$]):

$1/\epsilon$ implies
$$\begin{aligned} \epsilon x &= 1 \\ \epsilon^2 x &= \epsilon \\ 0 &= \epsilon \end{aligned}$$
impossible!

Double ($\epsilon^2 = 1$ [$\epsilon \neq 1$]):

$1/(1+\epsilon)$ implies
$$\begin{aligned} (1+\epsilon)x &= 1 \\ (1-\epsilon^2)x &= 1-\epsilon \\ 0 &= 1-\epsilon \end{aligned}$$
impossible!

Note: One can easily proceed with formal calculations in which ϵ is treated algebraically but never evaluated until the last step.
It appears all physical quantities involve ϵ^2.

The Circular Functions
(The Relativistic "Factors")

Let Θ be a real number.

$$
\begin{aligned}
\textbf{EXP } \Theta &\equiv \exp(\epsilon\Theta) \\
&= 1 + (\epsilon\Theta) + \frac{(\epsilon\Theta)^2}{2!} + \frac{(\epsilon\Theta)^3}{3!} + \frac{(\epsilon\Theta)^4}{4!} + \cdots \\
&= \left[1 + \frac{(\epsilon\Theta)^2}{2!} + \frac{(\epsilon\Theta)^4}{4!} + \cdots\right] \\
&\qquad + \left[(\epsilon\Theta) + \frac{(\epsilon\Theta)^3}{3!} + \frac{(\epsilon\Theta)^5}{5!} + \cdots\right] \\
&= \textbf{cosh } (\epsilon\Theta) + \textbf{sinh } (\epsilon\Theta) \\
&= \left[1 + \epsilon^2\frac{\Theta^2}{2!} + \epsilon^4\frac{\Theta^4}{4!} + \cdots\right] \\
&\qquad + \epsilon\left[\Theta + \epsilon^2\frac{\Theta^3}{3!} + \epsilon^4\frac{\Theta^5}{5!} + \cdots\right] \\
&= \textbf{COSH } \Theta + \epsilon \textbf{ SINH } \Theta
\end{aligned}
$$

$$
\epsilon \text{ TANH } \Theta \equiv \tanh(\epsilon\Theta) = \frac{\sinh(\epsilon\Theta)}{\cosh(\epsilon\Theta)} = \frac{\epsilon \text{ SINH } \Theta}{\text{COSH } \Theta}
$$

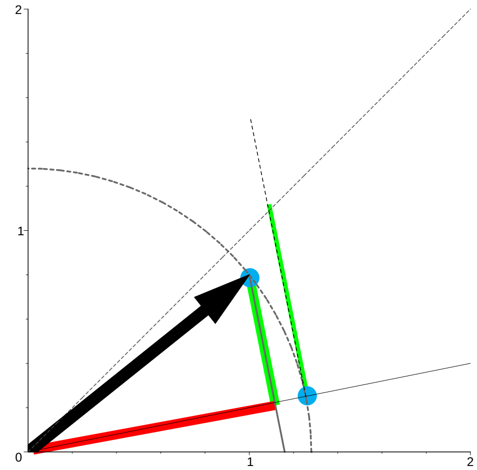
Algebraic Identities

$$
\begin{aligned}
1 &= \text{COSH}^2\Theta - \epsilon^2\text{SINH}^2\Theta \\
\text{TANH } (\Theta_1 + \Theta_2) &= \frac{\text{TANH } \Theta_1 + \text{TANH } \Theta_2}{1 + \epsilon^2 \text{ TANH } \Theta_1 \text{ TANH } \Theta_2} \\
\text{COSH } \Theta &= (1 - \epsilon^2 \text{ TANH}^2\Theta)^{-1/2}
\end{aligned}
$$

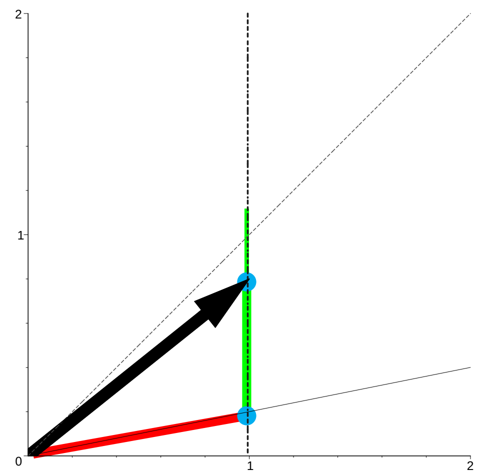
Differential Identities

$$
\begin{aligned}
\frac{d}{d\Theta}\text{EXP } \Theta &= \epsilon \text{ EXP } \Theta \\
\frac{d}{d\Theta}\text{COSH } \Theta &= \epsilon^2 \text{ SINH } \Theta \\
\frac{d}{d\Theta}\text{SINH } \Theta &= \text{COSH } \Theta \\
\frac{d}{d\Theta}\text{TANH } \Theta &= 1 - \epsilon^2 \text{ TANH }^2\Theta
\end{aligned}
$$

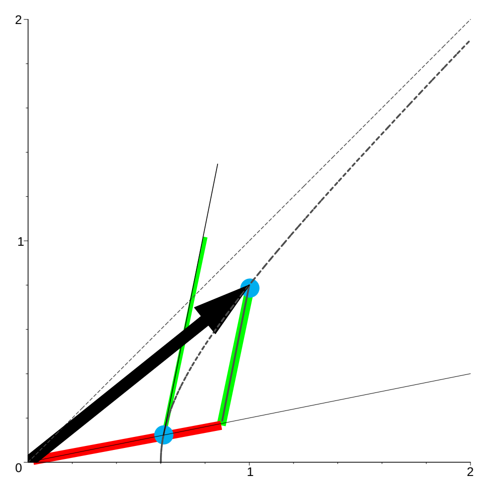
Every vector can be thought of as the HYPOTENUSE of some RIGHT triangle.



Euclidean decomposition of a vector.



Galilean decomposition of a vector.



Minkowskian decomposition of a vector.

Project the vector into components parallel and perpendicular to a given direction. "Drop the perpendicular" by constructing parallels to the tangent of the circle.

The Rotation
(Boost Transformation)

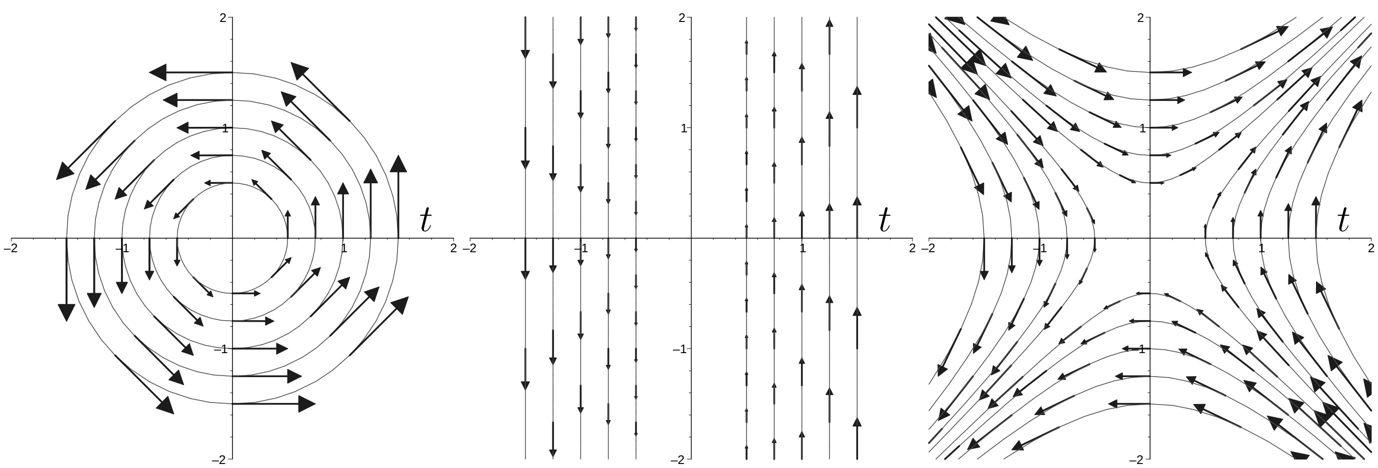
Consider a linear transformation $\vec{V}' = R(\Theta)\vec{V}$, where R satisfies:

$$\det R = 1 \qquad R(0) = I$$
$$R^T G R = G \qquad R(\Theta)R(\Phi) = R(\Theta + \Phi)$$

In terms of an orthogonal basis $\{\hat{t}, \hat{y}\}$ with metric $G = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon^2 \end{pmatrix}$, we find the linear transformation

$$\boxed{R(\Theta) = \begin{pmatrix} \cosh\Theta & \epsilon^2 \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix}}$$

is a "rotation" for that metric.

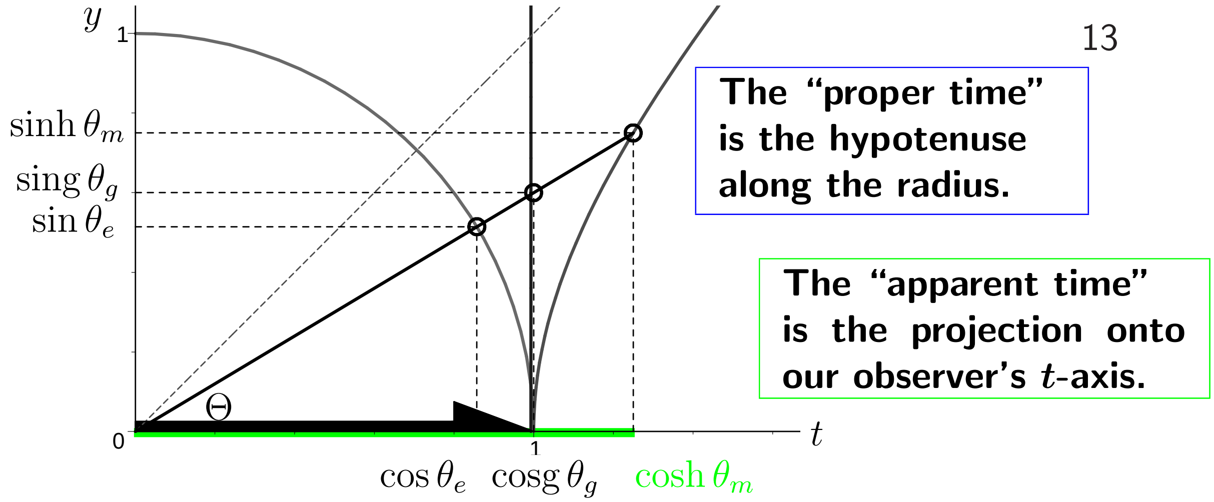


Euclidean Galilean Minkowskian

Eigenvectors of the Rotations
("Absolutes")

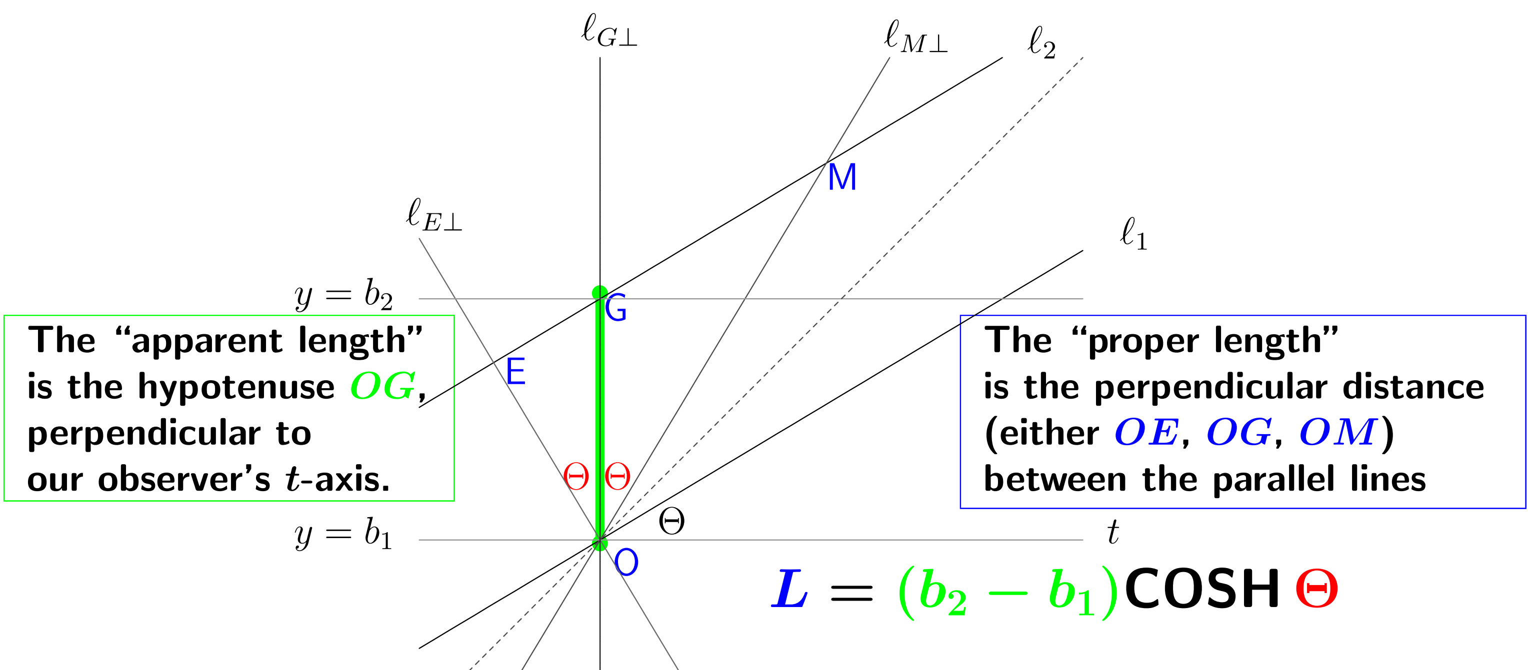
	eigenvalue	eigenvectors
EUC	$\cos\theta \pm i \sin\theta$ (actually, complex)	$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (better: invariant vector)
GAL	1 "absolute length"	$\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "absolute time"
MIN	$\cosh\theta \pm \sinh\theta = \exp(\pm\theta)$ $= \sqrt{\dfrac{1 \pm v}{1 \mp v}}$ Doppler-Bondi factor	$\hat{k} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\hat{l} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ "absolute speed of light"

Projection onto a Line
Proper-Time (and "Time-Dilation")



The arrow along the t-axis is a unit-vector in each geometry. Follow the arc along each "circle" to a line with slope v. Note that the corresponding unit vectors generally have different projections onto the t-axis.

The Distance between Parallels
Proper-Length (and "Length Contraction")

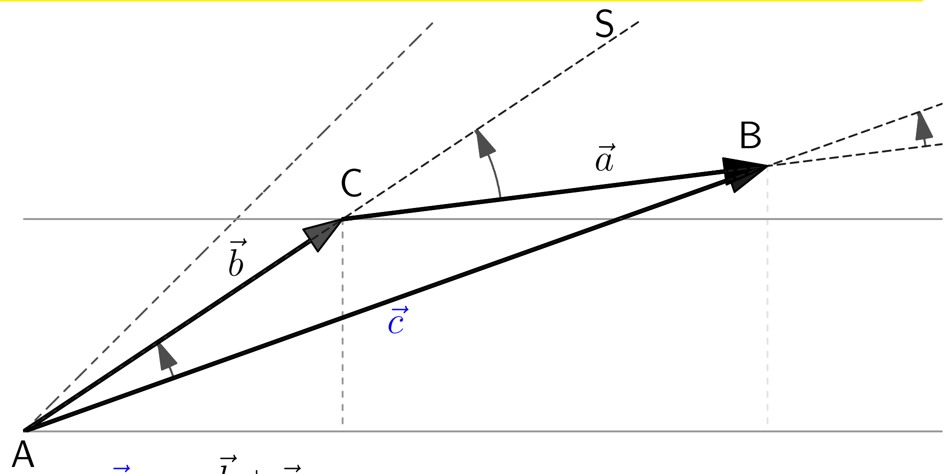


$$L = (b_2 - b_1)\,\mathrm{COSH}\,\Theta$$

Note that E, G, and M are "right angles" in their respective geometries. So, OG (being the side of the triangle opposite to the "right angle") is the hypotenuse in each geometry. (In Galilean relativity, the triangle is degenerate.)

(In special relativity, one sees the above relation in the form $\dfrac{L}{\mathrm{COSH}\,\Theta} = (b_2 - b_1)$.)

The Law of Cosines ("The Clock Effect")



$$\begin{aligned} \vec{c} &= \vec{b} + \vec{a} \\ \vec{c}\cdot\vec{c} &= \vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a} + 2\vec{b}\cdot\vec{a} \\ c^2 &= b^2 + a^2 + 2ba \text{ COSH } (m\angle BCS) \end{aligned}$$

In terms of the proper-time elapsed,

$$(t_{AB})^2 = (t_{AC})^2 + (t_{CB})^2 + 2t_{AC}t_{CB} \text{ COSH } (m\angle BCS)$$

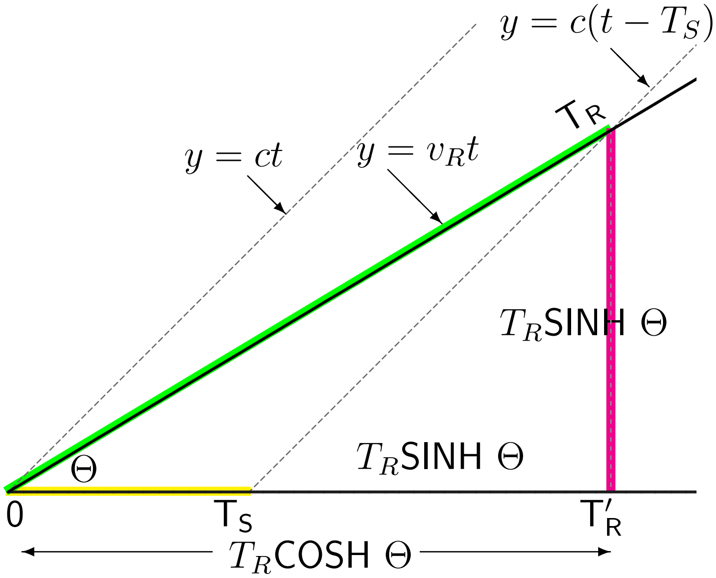
Comparing this with the identity

$$(t_{AC} + t_{CB})^2 = (t_{AC})^2 + (t_{CB})^2 + 2t_{AC}t_{CB}$$

and using the facts that $\cos\theta_e \leq 1$, $\operatorname{cosg}\theta_g = 1$, and $\cosh\theta_m \geq 1$,
the Law of Cosines implies the following relations:

$t_{AB} < t_{AC} + t_{CB}$	for $\epsilon^2 = -1$	"triangle inequality"
$t_{AB} = t_{AC} + t_{CB}$	for $\epsilon^2 = 0$	non-"clock effect"
$t_{AB} > t_{AC} + t_{CB}$	for $\epsilon^2 = +1$	"clock effect"

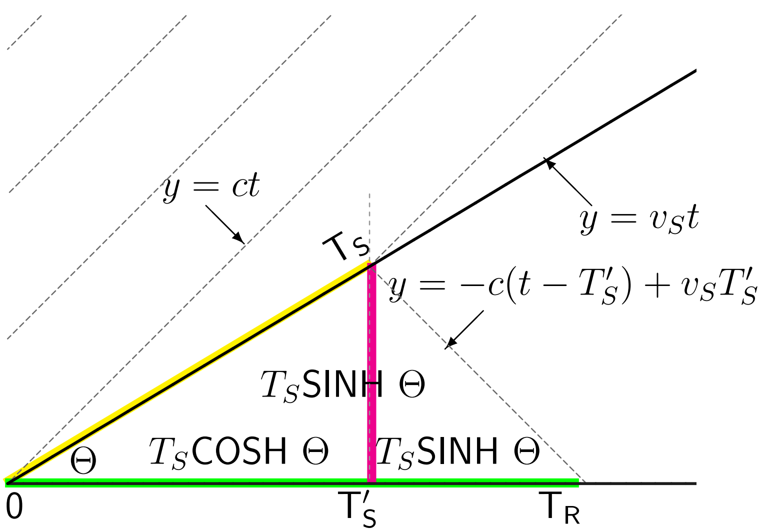
“The Doppler Effect”
a unified trigonometric derivation



Moving [Receding] Receiver

$$T_S = T_R(\text{COSH }\Theta - \text{SINH }\Theta)$$

$$\nu_R = \begin{cases} \nu_S(1 - \frac{v}{c}) & \text{Gal} \\ \nu_S\sqrt{\dfrac{1-\frac{v}{c}}{1+\frac{v}{c}}} & \text{Min} \end{cases}$$



Moving [Receding] Source

$$T_R = T_S(\text{COSH }\Theta + \text{SINH }\Theta)$$

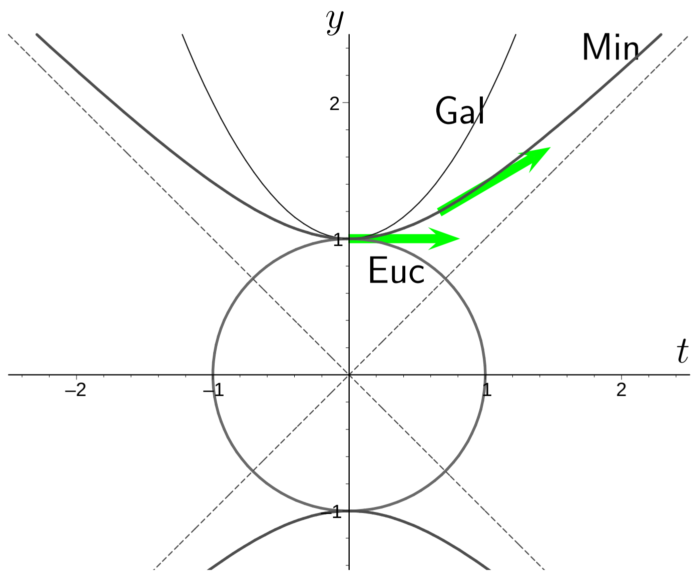
$$T_S = T_R\frac{1}{(\text{COSH }\Theta + \text{SINH }\Theta)}$$

$$\nu_R = \begin{cases} \nu_S\dfrac{1}{(1 + \frac{v}{c})} & \text{Gal} \\ \nu_S\sqrt{\dfrac{1-\frac{v}{c}}{1+\frac{v}{c}}} & \text{Min} \end{cases}$$

Note: In Minkowskian geometry ($\epsilon^2 = +1$),

$$\frac{1}{(\text{COSH }\Theta + \text{SINH }\Theta)} = (\text{COSH }\Theta - \text{SINH }\Theta).$$

The Curve of Constant Curvature
The Uniformly Accelerating Observer



The **curvature** ρ of a plane curve is a measure of how the **angle** ϕ of the **tangent vector** $\dot{y}$ changes with **arc-length** s along the curve. We will consider a timelike plane curve $y(t)$, i.e., a curve whose tangent is everywhere timelike.

The **acceleration** ρ of a worldline is a measure of how the **rapidity** ϕ of the **velocity vector** $\dot{y}$ changes with **proper-time** s along the curve.

$$\boldsymbol{\rho \equiv \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = \frac{\ddot{y}}{[1-\epsilon^2(\dot{y})^2]^{3/2}}}$$

We seek the **curve of constant curvature:** $\rho = a_0$:

If $\epsilon^2 \neq 0$,

$$(y-y_0)^2 - \frac{1}{\epsilon^2}(t-t_0)^2 = \frac{1}{\epsilon^4}{a_0}^{-2} \qquad \begin{cases} \text{if } \epsilon^2 = -1 & \textbf{circle} \\ \text{if } \epsilon^2 = +1 & \textbf{hyperbola} \end{cases}$$

If $\epsilon^2 = 0$,

$$y - y_0 = \frac{1}{2}a_0(t-t_0)^2 \qquad \{ \qquad \textbf{parabola}$$

"uniform acceleration" is an "invariant state of motion"

Trigonometric Identities
Transformations of spatial-velocity and spatial-acceleration

Consider an object whose spacetime position is specified by $\vec{y}(s)$, where s is the arc-length (proper time). Let an inertial-observer O_1 study this object and assign it coordinates $y_1(t_1)$. Similarly, assign $y_2(t_2)$ for an inertial-observer O_2. Let v_{21} be the invariant relative velocity of O_2 with respect to O_1:

$$v_{21} = \text{TANH }(\Theta_{20} - \Theta_{10}),$$

where the rapidities are measured with respect to some fiducial timelike axis (a third observer). It will be convenient to use the abbreviation

$$\Theta_{21} = \Theta_{20} - \Theta_{10}.$$

First, suppose that $\vec{y}$ is inertial.
If O_1 says that $\vec{y}$ moves with spatial velocity $\frac{dy_1}{dt_1} = v_{y1} =$ TANH Θ_{y1}, what does O_2 say? That is, express $\frac{dy_2}{dt_2} = v_{y2} = \text{TANH } \Theta_{y2}$ in terms of Θ_{y1} and Θ_{21}.

$$\begin{aligned} v_{y2} &= \text{TANH } \Theta_{y2} \\ &= \text{TANH } (\Theta_{y1} - \Theta_{21}) \\ &= \frac{\text{TANH } \Theta_{y1} - \text{TANH } \Theta_{21}}{1 - \epsilon^2 \text{TANH } \Theta_{y1} \text{TANH } \Theta_{21}} = \frac{v_{y1} - v_{21}}{1 - \epsilon^2 v_{y1} v_{21}} \end{aligned}$$

Suppose now that $\vec{y}$ is uniformly accelerated.
If O_1 says that $\vec{y}$ moves with spatial acceleration $\frac{d^2y_1}{dt_1{}^2} = a_{y1} = \rho_y \frac{1}{\textbf{COSH}^3 \Theta_{y1}}$, what does O_2 say? That is, express $\frac{d^2y_2}{dt_2{}^2} = a_{y2} = \rho_y \frac{1}{\text{COSH}^3 \Theta_{y2}}$ in terms of Θ_{y1} and Θ_{21}.

$$\begin{aligned} a_{y2} &= \frac{\rho_y}{\text{COSH}^3 \Theta_{y2}} \\ &= \frac{\rho_y}{\text{COSH}^3 (\Theta_{y1} - \Theta_{21})} \\ &= \frac{\rho_y}{(\text{COSH } \Theta_{y1} \text{COSH } \Theta_{21} - \epsilon^2 \text{SINH } \Theta_{y1} \text{SINH } \Theta_{21})^3} \\ &= \frac{\rho_y}{\text{COSH}^3 \Theta_{y1}} \frac{1}{\text{COSH}^3 \Theta_{21} (1 - \epsilon^2 \text{TANH } \Theta_{y1} \text{TANH } \Theta_{21})^3} = a_{y1} \left(\frac{\sqrt{1 - \epsilon^2 {v_{21}}^2}}{1 - \epsilon^2 v_{y1} v_{21}} \right)^3 \end{aligned}$$

Euclidean Postulate I
Simultaneity

Euclid I : "To draw a unique straight **"line"** from any point to any point."

Euclid V **(Playfair)** : "Given a **line**, and a **point** not on that **line**, there exists **only one line** through that **point** which **is parallel to** (i.e., **does not "intersect"**) the given **line**. *[This asserts the existence and uniqueness of a parallel to a given line through a given point.]*

"duality" in projective geometry exchanges points with lines, and so forth...

Euclid I **(dual-Playfair)** : "Given a **point**, and a **line** not through that **point**, there exists **no point** on that **line** which **cannot be joined (by an "ordinary" line) to** the given **point**. *[This asserts the nonexistence of a non-ordinary line from a given point.]*

Euclid I **(spacetime)** : "Given an **event**, and a **worldline** not experiencing that **event**, there exists **no event on that worldline** which **is not "timelike-related" to** the given **event**. *[This asserts the nonexistence of non-timelike-related events from a given event.]*

Loosely speaking, regard

(the spacetime interpretation of) Euclid I

as a statement concerning

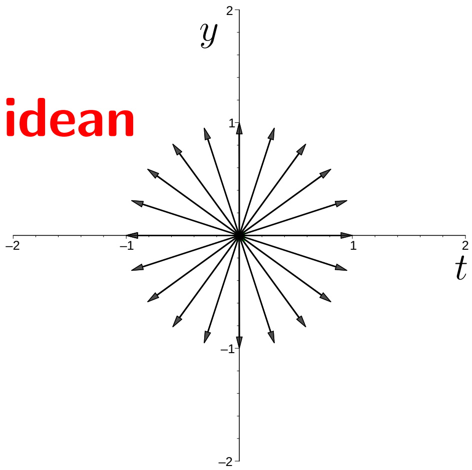
"simultaneity with distant events"

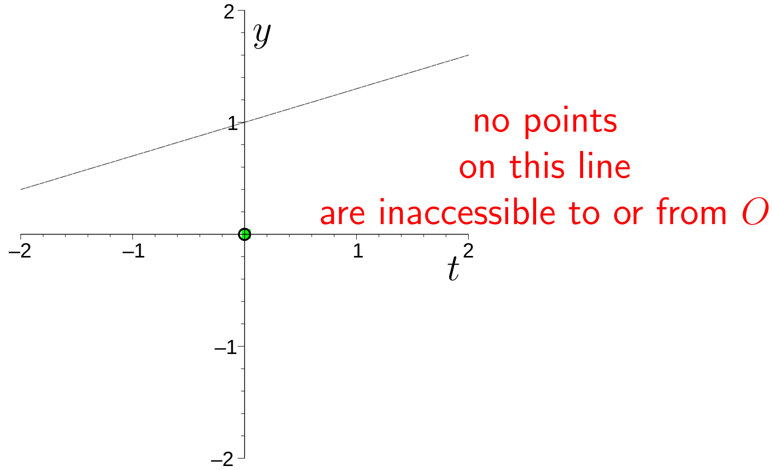
spacetime geometry	*an event (on a given distant worldline) that is simultaneous with our given event*
Euclidean	**does not exist**
Galilean	**exists and is unique**
Minkowskian	**exists and is not unique**

Spacetime Geometry is non-Euclidean

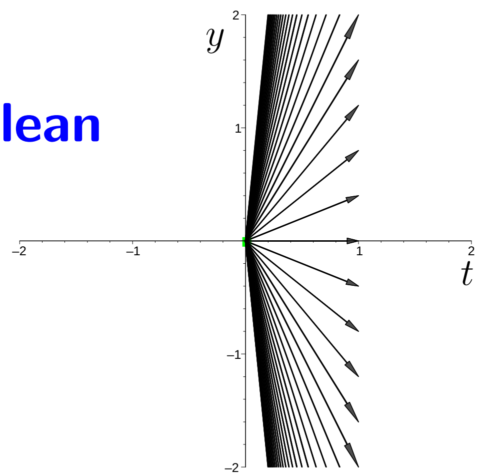
Consider a given point [event],
and a straight line [inertial worldline]
not through [experiencing] that point [event].

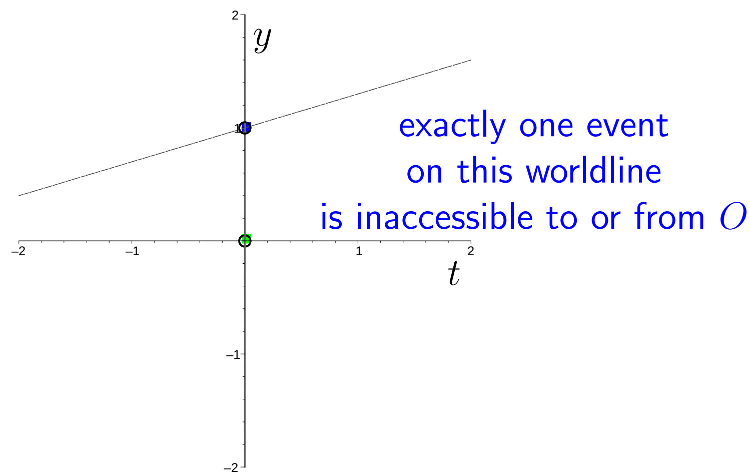
Euclidean



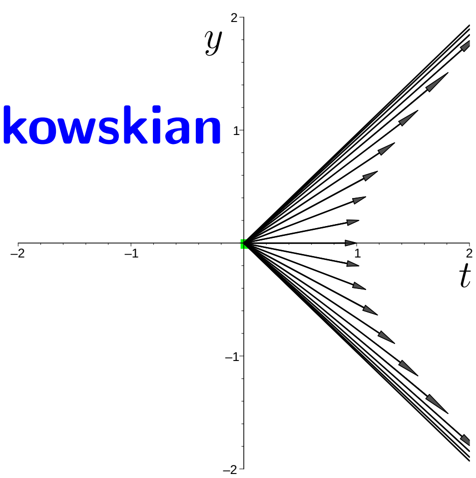


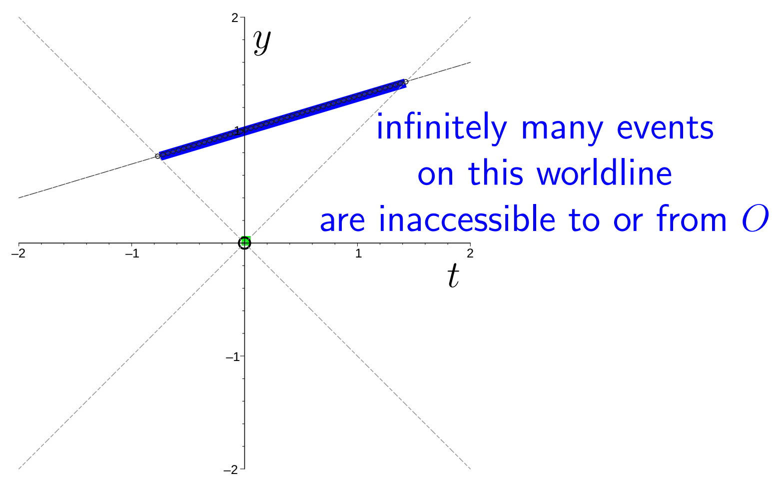
Galilean





Minkowskian





AAPT Topical Conference: Teaching General Relativity to Undergraduates
AAPT Summer Meeting
Syracuse, NY, July 20-21,22-26 2006

Spacetime Trigonometry and Analytic Geometry I: The Trilogy of the Surveyors

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I. INTRODUCTION

This is the first of a series of articles in which we expound a unified formalism for two-dimensional Euclidean space, Galilean spacetime,[1 2] and Minkowski spacetime rooted in the geometrical studies of Arthur Cayley and Felix Klein. Using techniques familiar from the analytic geometry and trigonometry of Euclidean space, we develop the corresponding analogues for Galilean and Minkowskian spacetimes and provide them with physical interpretations. This provides a new approach for teaching relativity and allows us to clarify many terms used in relativity.

Our presentation is primarily inspired by two works:

- I.M. Yaglom's *A Simple Non-Euclidean Geometry and Its Physical Basis*,[3]
  which is an insightful study of the geometry associated with the Galilean transformations,
- E.F. Taylor and J.A. Wheeler's *Spacetime Physics*,[4]
  which presents Special Relativity from a geometrical viewpoint.

II. THE TRILOGY OF THE SURVEYORS

(These passages were inspired by "The Parable of the Surveyors" in E.F. Taylor and J.A. Wheeler's *Spacetime Physics*.[5])

A. The Euclidean Surveyors

Once upon a time, a student of Euclid was asked to devise a method to survey an unexplored territory of the kingdom, a featureless plane stretching as far as the eye can see. So, he organized a team of surveyors and equipped each surveyor with a pair of ideal measuring tools. The first tool is an "odometer," or "rolling tape-measure," which measures the distance a surveyor has traveled along his path, in units of miles. The second tool is a very long ruler, calibrated in feet, which measures the perpendicular distance from his path to a distant point not his path.

He instructed the surveyors to begin at a common origin, O, then to travel in a straight line in all possible spatial directions in the plane. Each is instructed to mark the point with a flag when his tape-measure reads "t=1 mi." What is the locus of these points? Of course, the result of the experiment is that these points lie on a "circle with radius $R = 1$ mi." This circle, the Euclidean student declared, provides the basic calibration curve for measuring the separation between points on this plane. Indeed, later surveys of this territory would show that such a circle could be constructed from any origin and be extended to any radius.

In order to complete the survey of the plane, each surveyor is told to assign to each point a pair of coordinates (t, y), where t represents the displacement "along his path" (as measured by his tape-measure) and y represents the displacement "perpendicular to his path" (as measured by his ruler). What does "perpendicular to his path" mean? Geometrically, it means "along the tangent-line to the circle of radius t" at his point on that circle. Operationally, this means "along a line of points that he regards as having the same value of t."

When the radial surveyors' data were collected, it was noticed that for any given point P on the circle, there was general disagreement on its assigned t-displacements and on its assigned y-displacements from the origin O. That is, comparing the measurements from two radial surveyors U and U', we have

$$(t_P - t_O) \neq (t'_P - t'_O) \qquad \text{and} \qquad (y_P - y_O) \neq (y'_P - y'_O).$$

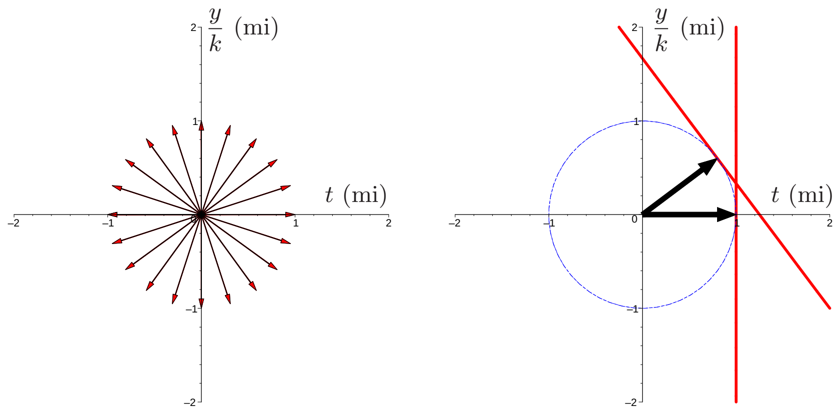

FIG. 1: The Euclidean surveyors define a unit circle, the basic calibration curve for measuring separations in space. Each surveyor operationally defines "perpendicular to his radial path" as "along his tangent-line to the circle."

However, there was unanimous agreement on the quadratic quantity $(t_P - t_O)^2 + ((y_P - y_O)/k)^2$:

$$(t_P - t_O)^2 + \left(\frac{y_P - y_O}{k}\right)^2 = (t'_P - t'_O)^2 + \left(\frac{y'_P - y'_O}{k}\right)^2,$$

where k is a unit-conversion constant,[6] which simplifies the arithmetic. The Euclideans called this quadratic quantity the "square-distance" from point O, which provides an invariant measure of the separation between points P and O. (See Fig. 1) Indeed, each radial surveyor used the same equation to describe the circle:

$$(t - t_O)^2 + \left(\frac{y - y_O}{k}\right)^2 = R^2.$$

In fact, by comparing the unlabeled plots from each radial surveyor, the Euclidean student could not distinguish among these radial surveyors. So, no radial surveyor is preferred over any other. However, when the labeling is accounted for, there is a way to relate the measurements of one radial surveyor with those of another.[7]

Having been successful at defining an invariant measure of separation between pairs of points, the Euclidean student now wished to define an invariant measure of separation between pairs of radial surveyor-paths through O. It seemed reasonable to define this measure using the signed arc-length S_e of the circle cut by those paths. However, this measure was deemed to be somewhat limited because the arc-length S_e depends on the radius R of the circle used to determine it. After a little analysis, it was observed that for a given pair of surveyor-paths, the corresponding values of S_e and R are in constant proportion. So, in order for the measure of separation to be independent of the size of the circle used, the Euclidean student defined a quantity called the signed "angle" by

$$\theta_e \equiv \left(\frac{S_e/k}{R}\right),$$

where the unit-conversion constant k is used to make this angle dimensionless.

The Euclidean student observed that, since the arc-length S_e is bounded, the range of the angle θ_e is bounded: $-\pi < \theta_e \leq \pi$, where π is a numerical constant that was determined to be approximately 3.14159. Since the arc-length is additive, the angle measure is also an additive quantity. That is, for radial surveyor-paths OA, OB, and OC from O through the respective points A, B, and C on the circle, we have

$$\theta_{e,AB} + \theta_{e,BC} = \theta_{e,AC},$$

where, for example, $\theta_{e,AB}$ is the signed-angle from OA to OB.

Many years later, another student of Euclid defined another invariant measure of separation between pairs of radial surveyor-paths in terms of the parallel and perpendicular projections, $(t - t_O)$ and $(y - y_O)$, respectively, of one surveyor path onto another surveyor path. This quantity, called the "slope," is defined by $v_e \equiv (y - y_O)/(t - t_O)$, which is expressed in units of miles/ft. This can be expressed in terms of the angle θ_e using

$$\frac{v_e}{k} \equiv \frac{(y - y_O)/k}{(t - t_O)} = \tan\left(\frac{S_e/k}{R}\right) = \tan(\theta_e),$$

where we have used the unit-conversion constant k since the trigonometric tangent function and its argument are dimensionless. Observe that the range of the slope is unbounded: $-\infty < v_e < \infty$ [in units of ft/mile]. Furthermore, unlike the angle measure, the slope is not an additive quantity:

$$v_{e,AB} + v_{e,BC} \neq v_{e,AC}.$$

Instead,

$$\frac{v_{e,AB} + v_{e,BC}}{1 - v_{e,AB} v_{e,BC}/k^2} = v_{e,AC}.$$

As a postscript to this story, the use of the unit-conversion constant k was eventually seen as an unnecessary nuisance in the arithmetic. By expressing "ruler measurements" for "y" (traditionally in units of feet) in the "odometer units" used by "t" (traditionally in units of miles), the need to use k was eliminated. With this advancement, the Euclidean results can be written more simply as:

$$(t - t_O)^2 + (y - y_O)^2 = R^2$$

$$\theta_e \equiv S_e/R \qquad\qquad \theta_{e,AB} + \theta_{e,BC} = \theta_{e,AC}$$

$$v_e \equiv \frac{(y - y_O)}{(t - t_O)} = \tan(S_e/R) = \tan\theta_e \qquad\qquad \frac{v_{e,AB} + v_{e,BC}}{1 - v_{e,AB} v_{e,BC}} = v_{e,AC}$$

B. The Galilean Surveyors

In the study of kinematics, one naturally draws "distance vs. time" graphs. This is a different kind of space, called "spacetime," whose points are called "events." Thus, a "distance vs. time" graph is sometimes called a "spacetime diagram." How might a student of Galileo proceed to survey spacetime? He decided to follow an analogue of the Euclidean procedure used in the preceding section.

He organized a team of "spacetime surveyors," henceforth, called "observers". As time elapses, each observer will trace out a path in spacetime called his "worldline." Those observers who travel inertially will trace out straight worldlines in spacetime.

Each observer is equipped with a pair of ideal measuring tools. The first tool is a "chronometer," or "wristwatch," which measures the interval of time that has elapsed along his worldline, in units of seconds. The second tool is a very long ruler, calibrated in meters, which measures the "[Galilean-]perpendicular" distance from his worldline to a distant event not on his worldline. [We will more fully define *this* notion of perpendicularity shortly.]

He instructed the observers to begin at a common event, O, then to travel inertially with all possible velocities along a common straight line in space. Each is instructed to mark the event with a firecracker explosion when his wristwatch reads "$t = 1$ second". What is the locus of these events on a spacetime diagram?

Due to the technological limitations of their day, the maximum speed attempted by the Galilean observers was a small fraction of the speed of light, c_{light}, which they knew to be finite. Nevertheless, they boldly extrapolated their experimental observations and reached the conclusion that these events lie on a vertical line in our spacetime diagram, which they might have called a "[future Galilean] circle with radius $R = 1$ second." This [Galilean] circle, the Galilean student declared, provides the basic calibration curve for measuring the separation between events in spacetime. Note that this declaration implicitly asserts that all finite speeds (including those faster than the speed of light, c_{light}) are theoretically attainable by observers!

In order to complete the survey of the spacetime, each observer is told to assign to each event a pair of coordinates (t, y), where t represents the temporal displacement "along his worldline" (as measured by his wristwatch) and y represents the spatial-displacement "perpendicular to his worldline" (as measured by his ruler). But what does "perpendicular to his worldline" mean here? Geometrically, following the Euclidean procedure, this means "along the tangent-line to the [Galilean] circle." (See Fig. 2.) Operationally, this means "along a line of events that he regards as having the same value of t." Physically, this means "along a line of events that he regards as simultaneous."

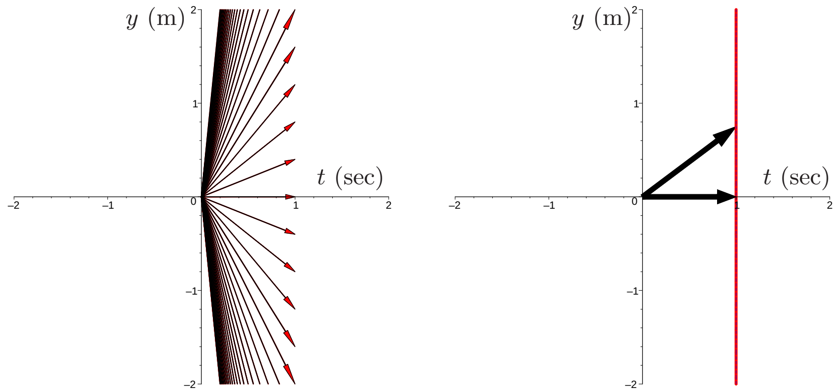

FIG. 2: The Galilean observers define a “[Galilean] unit circle,” the basic calibration curve for measuring intervals in spacetime. Each inertial observer operationally defines “perpendicular to his worldline” as “along his tangent-line to the [Galilean] circle.” Physically, the events that lie on a given line perpendicular to his worldline are simultaneous according to this observer.

When the observers’ data were collected, it was noticed that for any given event P on the Galilean circle, there was general disagreement on its assigned y-displacements from the origin O. That is, comparing the measurements from two inertial observers U and U', we have

$$(y_P - y_O) \neq (y'_P - y'_O).$$

However, there was unanimous agreement on its assigned t-displacements,

$$(t_P - t_O) = (t'_P - t'_O),$$

and, therefore, on the quadratic quantity $(t_P - t_O)^2$:

$$(t_P - t_O)^2 = (t'_P - t'_O)^2.$$

Inspired by the Euclidean observers, the Galileans decided to call this quadratic quantity the “[Galilean] square-interval,” which provides an invariant measure of the separation between events P and O. (See Fig. 2) Indeed, each inertial observer used the same equation to describe the [Galilean] circle:

$$(t - t_O)^2 = R^2.$$

In fact, by comparing the unlabeled plots from each inertial observer, the Galilean student could not distinguish among these inertial observers. So, no inertial observer is preferred over any other. However, when the labeling is accounted for, there is a way to relate the measurements of one inertial observer with those of another.[8]

The Galilean observers noted several geometrical features not seen by the Euclidean surveyors. First, about half of the spacetime could not be surveyed by the inertial observers starting at event O. Even if the experiment had been expanded to include inertial observers that would end up meeting at event O, the key feature noted was that *there was an inaccessible radial direction to or from event O* on their spacetime diagrams. This corresponds to a speed that was unattainable by the observers through event O. Moreover, all of these observers determine the same value for this unattainable speed: the universal constant $c_{max,g}$ whose value was experimentally extrapolated to be infinite. Second, for any given Galilean circle, the tangent-lines corresponding to different radii coincided. Physically, this was interpreted by saying that “simultaneity is absolute.” Moreover, for a given pair of events on a given tangent-line to the Galilean circle, all observers determine the same value for the spatial separation between that pair. Physically, this was interpreted by saying that “the measured length of an object [whose bounding worldlines pass through those events] is absolute.”

Following the Euclideans, the Galileans also wished to define an invariant measure of separation between pairs of inertial observer-worldlines through O. They defined that separation in terms of the [Galilean] arc-length S_g of a [Galilean] circle of radius R cut by those worldlines. They defined a quantity analogous to the Euclidean angle,

$$\theta_g \equiv \left(\frac{S_g/k_g}{R} \right),$$

called the "Galilean-angle," or, more physically, the "Galilean rapidity." Note that, in order for this angle to be dimensionless, one must introduce a unit-conversion constant k_g with units of meters/second. For ease of comparison with the Minkowskian case later, we will choose $k_g = c_{light}$. Since the arc-length S_g is unbounded, the rapidity θ_g is unbounded: $-\infty < \theta_g < \infty$. Since the [Galilean] arc-length is additive, the [Galilean] rapidity measure is an additive quantity. That is, for inertial observer-paths from O through events A, B, and C on the [Galilean] circle:

$$\theta_{g,AB} + \theta_{g,BC} = \theta_{g,AC}.$$

where, for example, $\theta_{g,AB}$ is the signed Galilean-rapidity from OA to OB.

They also defined another measure of separation between pairs of observer worldlines analogous to the slope, called the "velocity," by $v_g \equiv (y - y_O)/(t - t_O)$, where $(t - t_O)$ is the elapsed-time and $(y - y_O)$ is the spatial-displacement from event O. The velocity is expressed in units of meters/second. The Galilean observers were pleasantly surprised to find that the Galilean-rapidity is directly-proportional to the velocity

$$\frac{v_g}{k_g} \equiv \frac{(y - y_O)/k_g}{(t - t_O)} = \left(\frac{S_g/k_g}{R}\right) = \theta_g,$$

differing only by a unit-conversion constant. Thus, the velocity is also unbounded: $-\infty < v_g < \infty$ [in units of meters/second], as well as additive

$$v_{g,AB} + v_{g,BC} = v_{g,AC}.$$

This can be a great convenience for doing calculations in Galilean relativity. In particular, with the emphasis on velocity, there is little need to be concerned with [Galilean] rapidity and the unit-conversion constant k_g. However, the failure to recognize the distinction between [Galilean] rapidity and velocity will become one source of confusion for Galileans who will try to understand more modern models of spacetime.

C. The Minkowskian Surveyors

Consider now a technologically-advanced team of inertial observers, who can now attempt speeds comparable to c_{light}. If the same experiment were repeated, what is the locus of these events on a spacetime diagram?

Surprisingly [to the Galileans], the result of such an experiment[9] is that these events lie not on a straight line, but on a hyperbola in spacetime, which we call a "[future Minkowskian] circle with radius $R = 1$ second." This, of course, is consistent with the experimental results obtained by the Galilean observers since, for the small range of slow velocities that they attempted, the line is a good approximation to the hyperbola.

In the assignment of coordinates (t, y), what does "perpendicular to his worldline" mean here? Geometrically, this still means "along the tangent-line to the [Minkowskian] circle." Operationally, this still means "along a line of events that he regards as having the same value of t." Physically, this still means "along a line of events that he regards as simultaneous."

When the observers' data were collected, it was noticed that for any given event P on the Minkowskian circle, there was general disagreement on its assigned t-displacements and on its assigned y-displacements. That is, comparing the measurements from two inertial observers U and U', we have

$$(t_P - t_O) \neq (t'_P - t'_O) \qquad \text{and} \qquad (y_P - y_O) \neq (y'_P - y'_O).$$

However, there was unanimous agreement on the quadratic quantity $(t_P - t_O)^2 - ((y_P - y_O)/c_{light})^2$, i.e.,

$$\left(t_P - t_O\right)^2 - \left(\frac{y_P - y_O}{c_{light}}\right)^2 = (t'_P - t'_O)^2 - \left(\frac{y'_P - y'_O}{c_{light}}\right)^2,$$

where, at this stage, c_{light} is a unit-conversion constant[10] that we will call k_m. The Minkowskians decided to call this quadratic quantity the "[Minkowskian] square-interval," which provides an invariant measure of the separation between events P and O. Indeed, when the surveyors' plots were compared, it was also realized that all surveyors describe the same set of events on their Minkowskian circles

$$(t - t_O)^2 - \left(\frac{y - y_O}{c_{light}}\right)^2 = R^2.$$

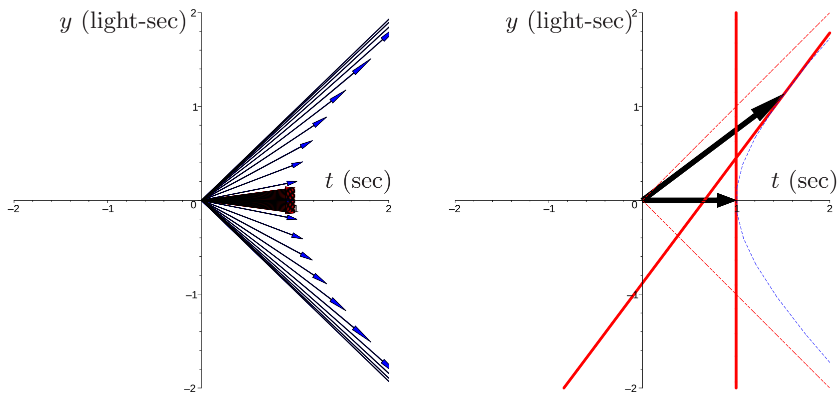

FIG. 3: The Minkowskian observers define a "[Minkowskian] unit circle," the basic calibration curve for measuring intervals in spacetime. (For comparison, we have included the efforts of the Galilean observers for their range of velocities. Note the units on the spatial axis.) Each inertial observer operationally defines "perpendicular to his worldline" as "along his tangent-line to the [Minkowskian] circle." Physically, the events that lie on a given line perpendicular to his worldline are simultaneous according to this observer.

In fact, by comparing the unlabeled plots from each inertial observer, the Minkowskian student could not distinguish among these inertial observers. So, no inertial observer is preferred over any other. However, when the labeling is accounted for, there is a way to relate the measurements of one inertial observer with those of another.[11]

The Minkowskian observers noted several geometrical features in conflict with those seen by the Galilean observers. First, *there were infinitely many inaccessible radial directions to or from event O* on their spacetime diagrams, which correspond to infinitely many speeds unattainable by the observers through event O. Moreover, all observers determine the same value for the least upper bound of the attainable speeds: the universal constant $c_{max,m}$ whose numerical value is measured to be equal to the speed of light c_{light}. (Note that the speed of light c_{light} plays two roles, one as a unit-conversion constant and the other as a maximum signal speed. We will emphasize this point in the next section.) Second, for a given Minkowskian circle, the tangent-lines corresponding to different radii of the same circle did not coincide. Physically, this was interpreted to mean that "simultaneity is, in fact, not absolute." Furthermore, the measured length of an object is not absolute.

The Minkowskian observers also wished to define a measure of separation between pairs of concurrent inertial observer-worldlines. Following the Euclidean and Galilean procedure, they defined the separation in terms of the arc-length S_m of a [Minkowskian] circle of radius R cut by those worldlines. The "Minkowski-angle", also known as the "rapidity," was defined by

$$\theta_m \equiv \left(\frac{S_m/k_m}{R}\right),$$

and the velocity by $v_m \equiv (y - y_O)/(t - t_O)$. In this case, however, the velocity and the rapidity are related by

$$\frac{v_m}{k_m} \equiv \frac{(y - y_O)/k_m}{(t - t_O)} = \tanh\left(\frac{S_m/k_m}{R}\right) = \tanh\left(\theta_m\right),$$

Observe that the range of the rapidity is unbounded: $-\infty < \theta_m < \infty$, whereas the velocity is bounded: $-c_{max,m} < v_m < c_{max,m}$. Furthermore, unlike the rapidity measure, the velocity is not an additive quantity:

$$v_{m,AB} + v_{m,BC} \neq v_{m,AC}.$$

Instead,

$$\frac{v_{m,AB} + v_{m,BC}}{1 + v_{m,AB}v_{m,BC}/k_m^2} = v_{m,AC}.$$

These results are also in conflict with the results obtained by the Galilean observers.

D. An observation about c_{light}

This trilogy suggests a similarity among these three geometries, which we will develop more fully in a subsequent article. Here, we begin to make precise those analogies by focusing on the line-elements of the three geometries.

Consider two infinitesimally-close points[12] O and P in each of two-dimensional Euclidean space, Galilean spacetime, and Minkowski spacetime. Let us write down the line-elements[13] or "infinitesimal square-intervals" for the three geometries:

$$(ds^2)_{Euc} = (dt)^2 + \left(\frac{dy}{k}\right)^2 \tag{1a}$$

$$(ds^2)_{Gal} = (dt)^2 \tag{1b}$$

$$(ds^2)_{Min} = (dt)^2 - \left(\frac{dy}{c_{light}}\right)^2, \tag{1c}$$

where, in our spacetime diagram, dt is the displacement from O to P along the horizontal axis and dy is the displacement from O to P along with the vertical axis. In order to facilitate the comparison of the three geometries, let us express the Euclidean measure of distance between two points in terms of the time it takes light to travel between them. Thus,

$$(ds^2)_{Euc} = (dt)^2 + \left(\frac{dy}{c_{light}}\right)^2,$$

where $(ds^2)_{Euc}$ is in units of square-seconds, (dt) is in units of seconds and (dy) is in distance units of light-seconds. We emphasize that, here, c_{light} plays the role of a convenient unit-conversion constant between the coordinates t and y. In this role, it has no physical significance.

Now, let us write

$$(ds^2)_{Euc} = (dt)^2 - (-1)\left(\frac{dy}{c_{light}}\right)^2 \tag{2a}$$

$$(ds^2)_{Gal} = (dt)^2 - (0)\left(\frac{dy}{c_{light}}\right)^2 \tag{2b}$$

$$(ds^2)_{Min} = (dt)^2 - (+1)\left(\frac{dy}{c_{light}}\right)^2, \tag{2c}$$

or in a unified way as

$$dS^2 = (ds^2)_{\epsilon} = (dt)^2 - \epsilon^2\left(\frac{dy}{c_{light}}\right)^2, \tag{3}$$

where ϵ^2 is a dimensionless quantity we will call the "indicator," which can take the value -1, 0, or 1, corresponding to the Euclidean, Galilean, and Minkowskian cases, respectively. [We will defer the discussion of the indeterminate ϵ for a later section.] Physically, the indicator can be interpreted as

$$\epsilon^2 \equiv \left(\frac{c_{light}}{c_{max}}\right)^2 \tag{4}$$

where c_{max} is the maximum signal speed of the particular spacetime theory. For the Minkowskian case, we have $c_{max} = c_{light}$, which can be inferred from Einstein's second postulate. For the Galilean case, we have $c_{max} = \infty$ since there is no upper bound on the speed of signal transmission in Galilean relativity. For consistency, if we were to interpret the Euclidean case as a space*time* theory, we would say $c_{max} = ic_{light}$, where i is the complex number $\sqrt{-1}$.[14]

Recognizing the role of c_{light} as merely a conversion factor in these equations, it is now convenient to redefine "y/c_{light}" as "y" (which is now measured in seconds) so that we can write

$$dS^2 = (ds^2)_{\epsilon} = dt^2 - \epsilon^2 dy^2, \tag{5}$$

where t and y are real-valued. Henceforth, we will use words and symbols in UPPER-CASE letters to indicate that the represented quantity has a dependence on the indicator ϵ^2. We will do this when emphasis is required. This formalism will allow us to discuss aspects of the three geometries and their physical interpretations in a unified way.

E. Postulates for Relativity

Consider the following formulation of the two postulates[15][16][17][18][19] for "relativity":

1. The laws of physics are the same for all inertial observers.

2. There is a real-valued maximum signal speed, c_{max}, and it is the same for all observers.

Special relativity declares the maximum signal speed to be the speed of light $c_{light} = 2.99792458 \times 10^8$ m/s. In terms of the indicator (4), special relativity corresponds to $\epsilon^2 = 1$. Galilean relativity, however, implicitly declares the maximum signal speed to be infinite. Thus, Galilean relativity corresponds to $\epsilon^2 = 0$.

In a subsequent article, we will more fully develop a unified presentation for Euclidean space, Galilean spacetime, and Minkowskian spacetime.

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[2] R. Geroch, *General Relativity from A to B* (The University of Chicago Press, Chicago, 1978); J. Ehlers, "The Nature and Structure of Spacetime," and A. Trautman, "Theory of Gravitation" in *The Physicist's Conception of Nature*, edited by J. Mehra (D. Reidel Publ, Dordrecht, Holland, 1973); R. Penrose, "Structure of Space-Time," in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by C.M. DeWitt and J.A. Wheeler (Benjamin, New York, 1967); M. Heller and D.J. Raine, *The Science of Space-Time* (Pachart Publishing House, Tucson, Arizona, 1981).

[3] I.M. Yaglom, *Ibid.*

[4] E.F. Taylor and J.A. Wheeler, *Spacetime Physics* (W.H. Freeman, New York, 1966).

[5] *Ibid.*

[6] Since t is measured in miles and y is measured in feet, the constant $k = 5280$ ft/mi.

[7] These are called rotations, which comprise a particular type of Euclidean transformation.

[8] These are called Galilean boosts, which comprise a particular type of Galilean transformation.

[9] D. Frisch and J. Smith, "Measurement of the Relativistic Time Dilation Using Mesons," Am. J. Phy. **31** 342 (1963).

[10] Since t is measured in seconds and y is measured in meters, the constant $c_{light} = 2.99792458 \times 10^8$ m/s.

[11] These are called Lorentzian boosts, which comprise a particular type of Lorentz transformation.

[12] We use "point" and "event" interchangeably.

[13] Note our signature conventions. Note also that, in most of the literature, the Galilean line-element is taken to be the spatial line-element $(ds^2)_{Gal} = (dy)^2$.

[14] H. Minkowski, "Space and Time" (1909) in *The Principle of Relativity* by H.A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, (Dover Publications, New York, 1923).
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