Spacetime Trigonometry:

a Cayley-Klein Geomety
approach to
Special and General Relativity

Rob Salgado

UW-La Crosse
Dept. of Physics

outline

- motivations

- the Cayley-Klein Geometries (various starting points)

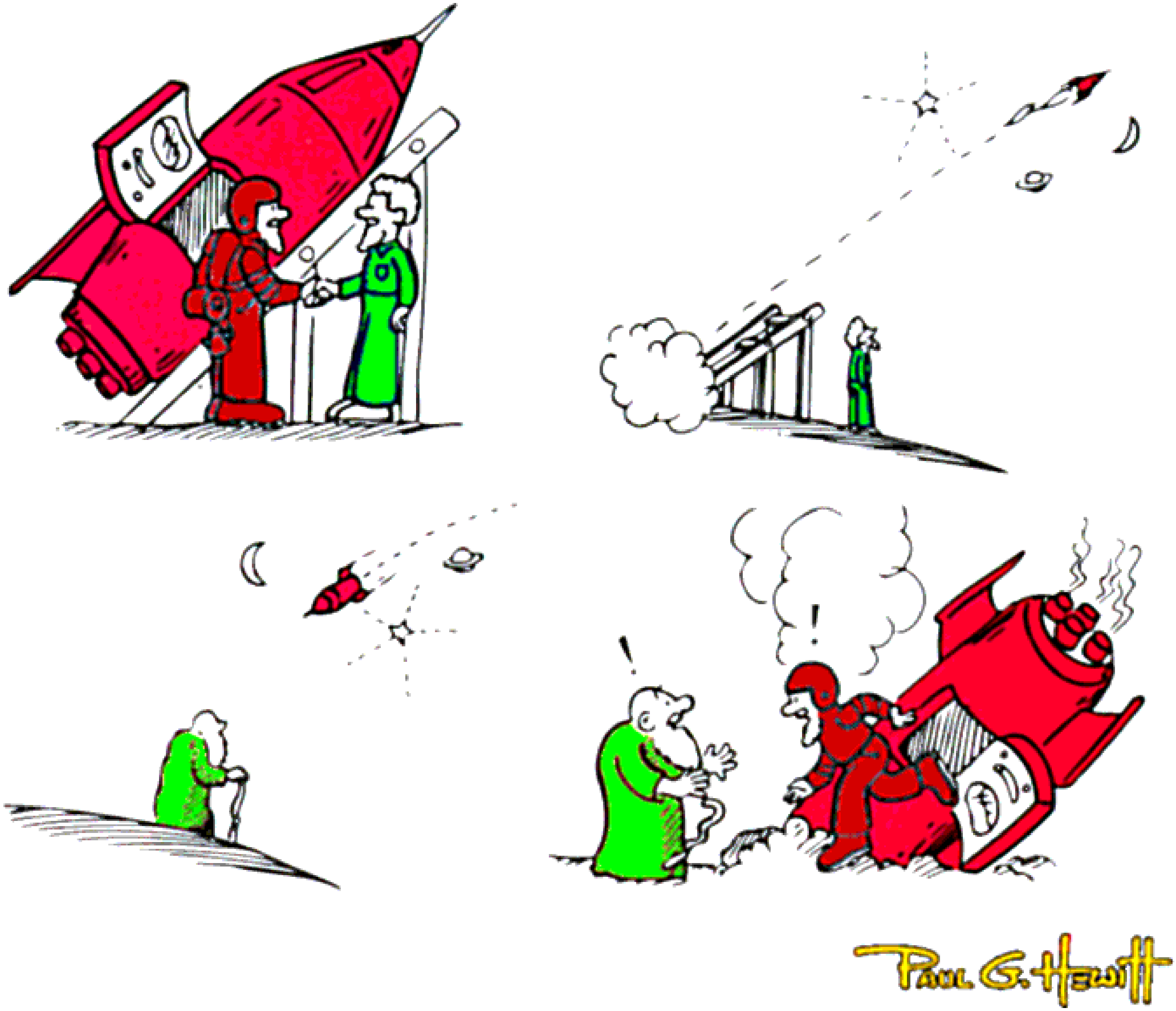
- Relativity (Trilogy of the Surveyors)
- development of **"Spacetime Trigonometry"**
  as a unified approach to the geometry of
  Galilean and Special Relativity

- [affine] Cayley-Klein Geometries (tour of more starting points)

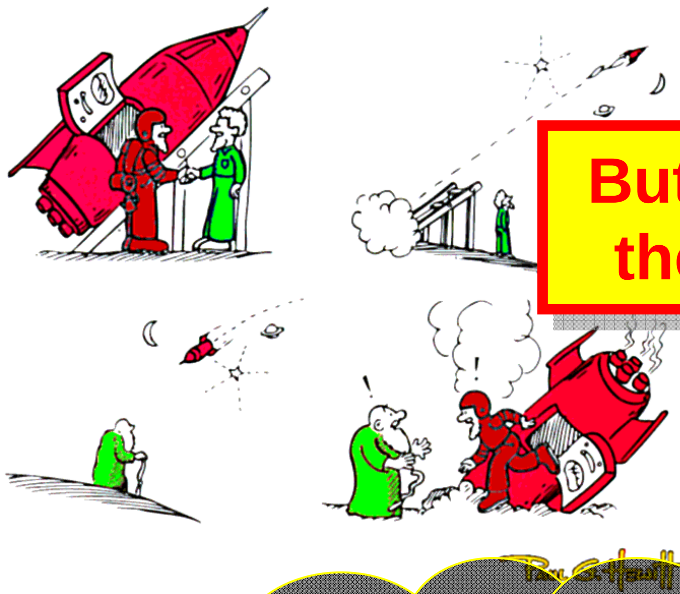
- **Spacetime Trigonometry**:
  "geometry of the Galilean spacetime
  as a bridge to Special Relativity"
  (How can some of the ideas be introduced to
  a physics student without all of the machinery that is available?

an infamous puzzle

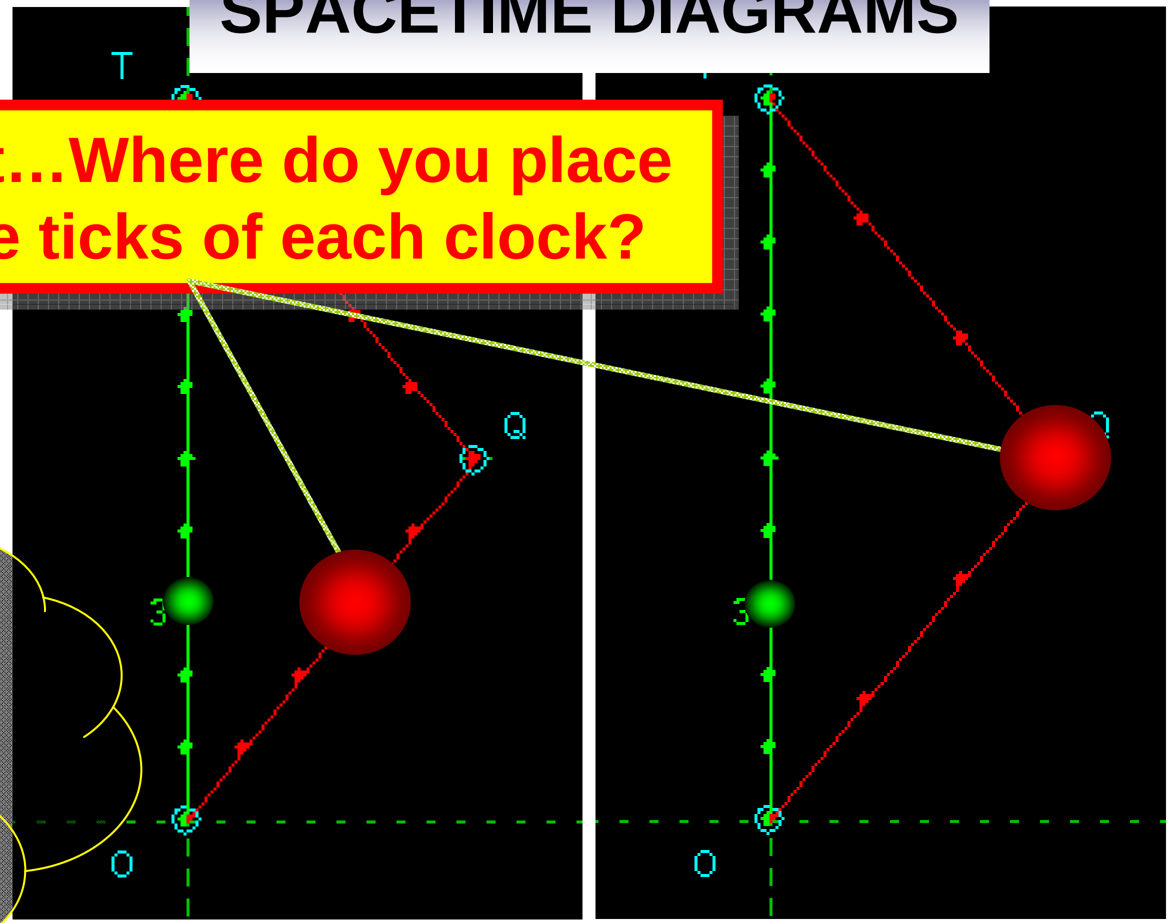
The Clock Effect / Twin Paradox



The Clock Effect / Twin Paradox



SPACETIME DIAGRAMS



But…Where do you place the ticks of each clock?

Spacetime Diagrams are the best way to analyze and interpret Relativity.

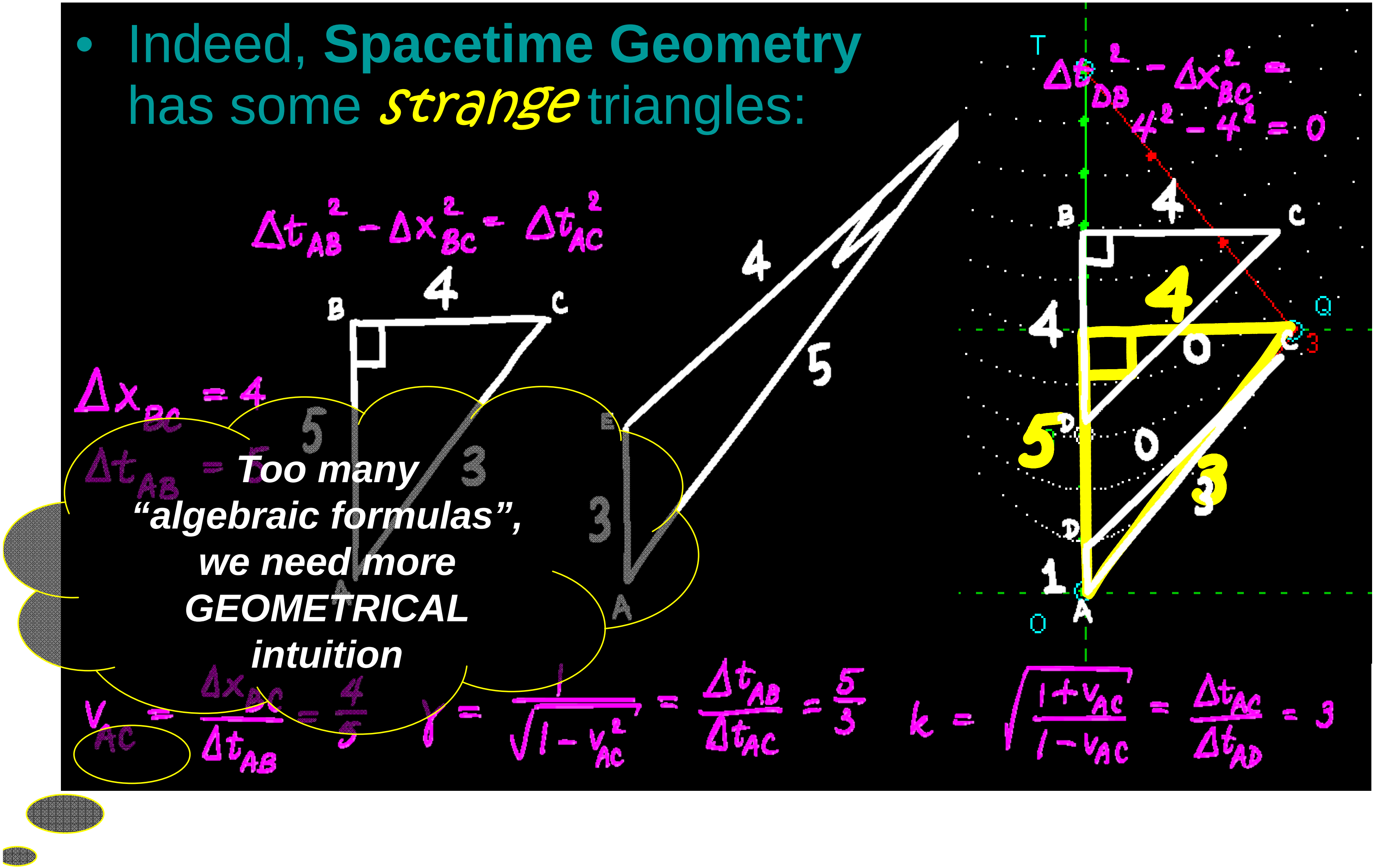
"common sense"

Galilean Relativity

Special Relativity

t runs upwards

- Indeed, **Spacetime Geometry** has some *strange* triangles:



Spacetime Trigonometry

GOAL:

Teach relativity by developing *geometric* intuition about spacetime.

HOW?

Exploit the *trigonometric* analogies

- Euclidean space
- Galilean spacetime
- Einstein-Minkowski spacetime

can its conceptual aspects be slipped into introductory physics??

EUCLIDEAN rotation

$$t' = (\cos\theta)t + (-\sin\theta)y \qquad t' = \left(\frac{1}{\sqrt{1+v^2}}\right)t + \left(\frac{-v}{\sqrt{1+v^2}}\right)y$$

$$y' = (\sin\theta)t + (\ \ \cos\theta)y \qquad y' = \left(\frac{v}{\sqrt{1+v^2}}\right)t + \left(\frac{1}{\sqrt{1+v^2}}\right)y$$

where $v = \tan\theta$.

$$\epsilon^2 = -1$$

GALILEAN boost

$$t' = (\textbf{cosg}\ \theta)t + (0\textbf{sing}\ \theta)y \qquad t' = \left(\ 1\ \right)t$$

$$y' = (\textbf{sing}\ \theta)t + (\ \ \textbf{cosg}\ \theta)y \qquad y' = \left(\ v\ \right)t + \left(\ 1\ \right)y$$

where $v = \textbf{tang}\ \theta$.

Yaglom defines $\textbf{cosg}\ \theta \equiv 1$, $\textbf{sing}\ \theta \equiv \theta$ so that $\textbf{tang}\ \theta \equiv \frac{\textbf{sing}\ \theta}{\textbf{cosg}\ \theta} \equiv \theta$.

$$\epsilon^2 = 0$$

MINKOWSKIAN boost

$$t' = (\cosh\theta)t + (+\sinh\theta)y \qquad t' = \left(\frac{1}{\sqrt{1-v^2}}\right)t + \left(\frac{v}{\sqrt{1-v^2}}\right)y$$

$$y' = (\sinh\theta)t + (\ \ \cosh\theta)y \qquad y' = \left(\frac{v}{\sqrt{1-v^2}}\right)t + \left(\frac{1}{\sqrt{1-v^2}}\right)y$$

where $v = \tanh\theta$.

$$\epsilon^2 = +1$$

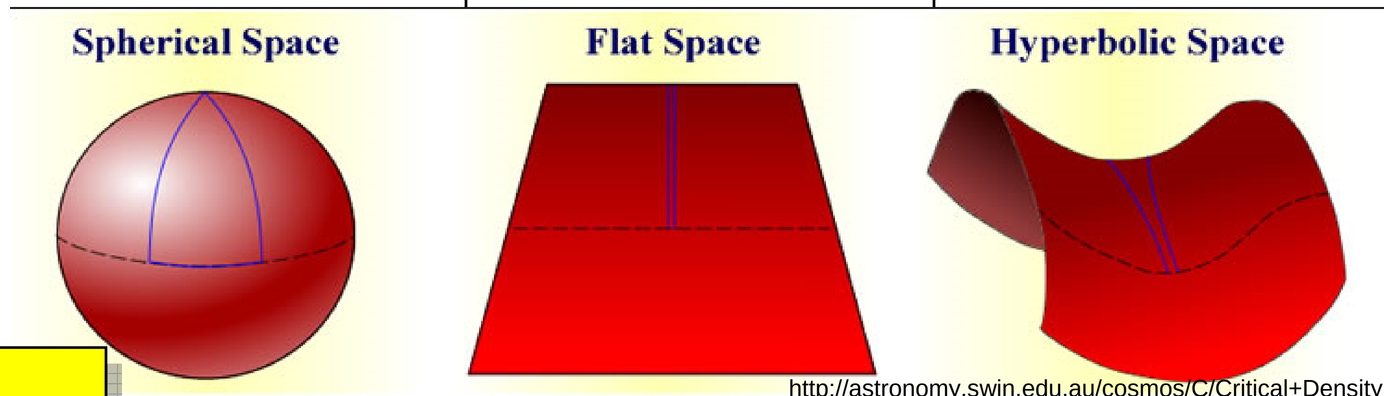
$$R(\Theta) = \begin{pmatrix} \text{COSH } \Theta & \epsilon^2\text{SINH } \Theta \\ \text{SINH } \Theta & \text{COSH } \Theta \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon^2 \end{pmatrix}$$

But first….**the classical Geometries**

(Cayley-Klein) measure of **Distance between Points**

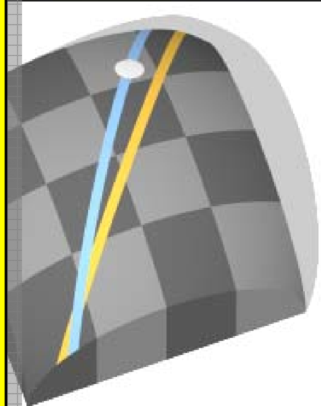
“elliptic”	“parabolic”	“hyperbolic”
Elliptic	**Euclidean**	**Hyperbolic**

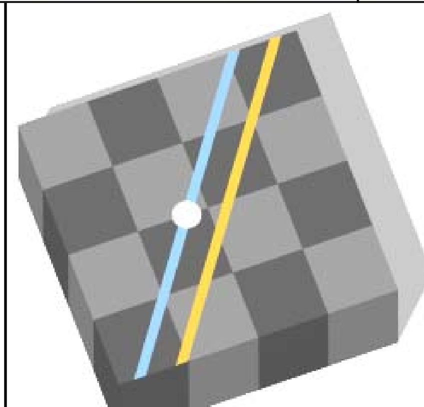
Initially-parallel lines…

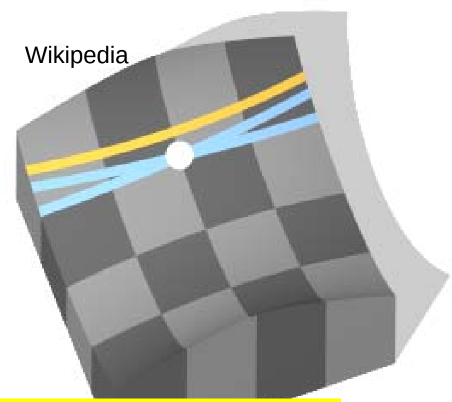


http://astronomy.swin.edu.au/cosmos/C/Critical+Density

EUCLID’s FIFTH (Playfair)
Given a line and
a point not on that line,
there exists
precisely one line
through that point
which does not intersect
(i.e., ``**is parallel to**")
the given line.


Zero parallels

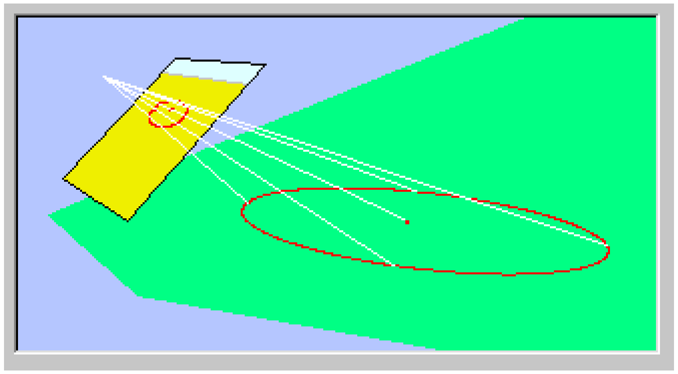

One parallel

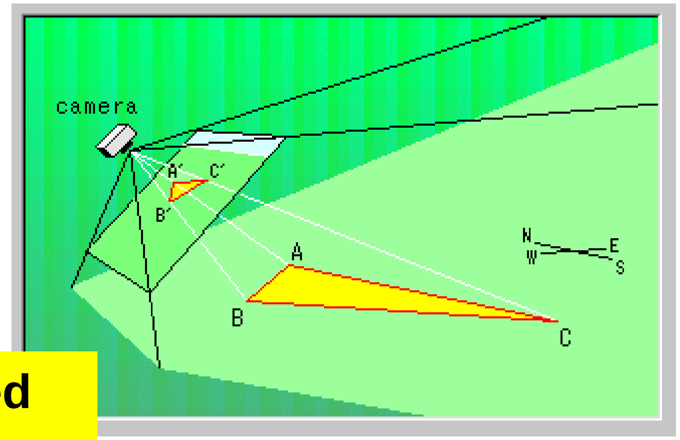


Infinitely many parallels

the **Cayley-Klein Geometries**

Sommerville uses **"duality"** between points and lines





Projective Geometry:

the geometry of perspective

Duality:

symmetry between points and lines

Any two distinct <u>points</u> are incident with exactly one <u>line</u>.

Joined by

Any two distinct <u>lines</u> are incident with exactly one <u>point</u>.

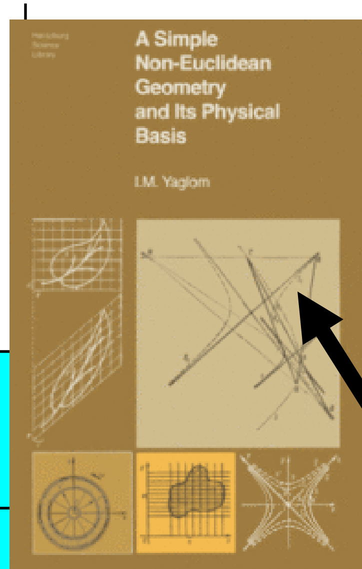
Meet at

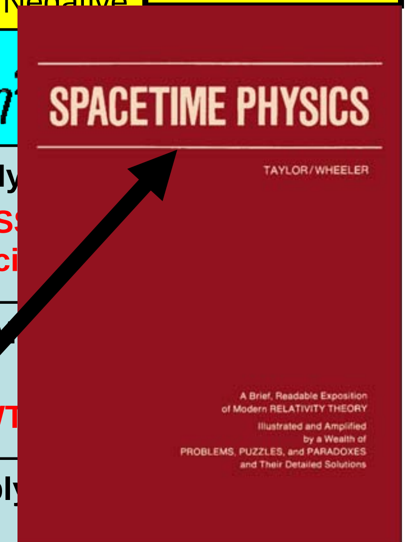
the **Cayley-Klein Geometries**

Sommerville uses **"duality"** between points and lines

measure of **Angle between Lines**		measure of **Distance between Points**: "elliptic"	"parabolic"	"hyperbolic"
	"elliptic"	**Elliptic**	**Euclidean**	**Hyperbolic**
	"parabolic"	**co-Euclidean**	**doubly-Parabolic**	**co-Minkowskian**
	"hyperbolic"	**co-Hyperbolic**	**Minkowskian**	**doubly-Hyperbolic**

		measure of **Distance between Points**		
		elliptic	parabolic	hyperbolic
		[Initially parallel lines…]	**intrinsic Curvature**	
		Positive	Zero	Negative
measure of **Angle between Lines**: elliptic	**Metric Signature** $(+,+)$	**Elliptic**	**Euclidean**	**Hyperbolic** **MASS-SHELL in Special Relativity**
parabolic	$(+,0)$	**co-Euclidean** **ANTI-NEWTON-HOOKE**	**doubly-Parabolic** **GALILEAN RELATIVITY**	**co-Minkowskian** **NEWTON-HOOKE**
hyperbolic	$(+,-)$	**co-Hyperbolic** **ANTI-DE-SITTER**	**Minkowskian** **SPECIAL RELATIVITY**	**doubly-Hyperbolic** **DE-SITTER**





			easure of **Distance between Points**		
				parabolic	hyperbolic
				intrinsic Curvature k	
				Zero	Negative
				$\eta^2 = 0$	η
measure of **Angle between Lines**: elliptic	**Metric Signature** (+,+)	$\epsilon^2 = -1$		**Euclidean**	**Hy** MAS Speci
parabolic	(+,0)	$\epsilon^2 = 0$	ANTI- NEWTON-HOOKE	**doubly-Parabolic** GALILEAN RELATIVITY	**co-M** NEWT
hyperbolic	(+,−)	$\epsilon^2 = +1$	**co-Hyperbolic** ANTI- DE-SITTER	**Minkowskian** SPECIAL RELATIVITY	**doubl** DE-SITTER

$$k = -\eta^2$$

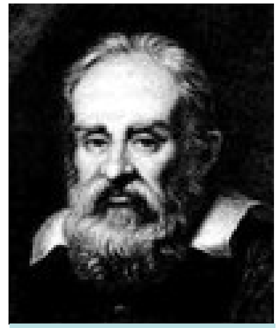
$$(+, -\epsilon^2)$$

$$
\begin{aligned}
ds^2 &= g_{ab}\, dx^a dx^b \\
&= \frac{(1+\eta^2\epsilon^2 y^2)dt^2 - (1-\eta^2 t^2)\epsilon^2 dy^2 - 2\eta^2\epsilon^2 ty\ dt\ dy}{(1-\eta^2(t^2-\epsilon^2 y^2))^2}
\end{aligned}
$$

$$\boxed{ds^2 = dt^2 - \epsilon^2 dy^2 \qquad (\eta^2 = 0)}$$

PHYSICS: Trilogy of the Surveyors

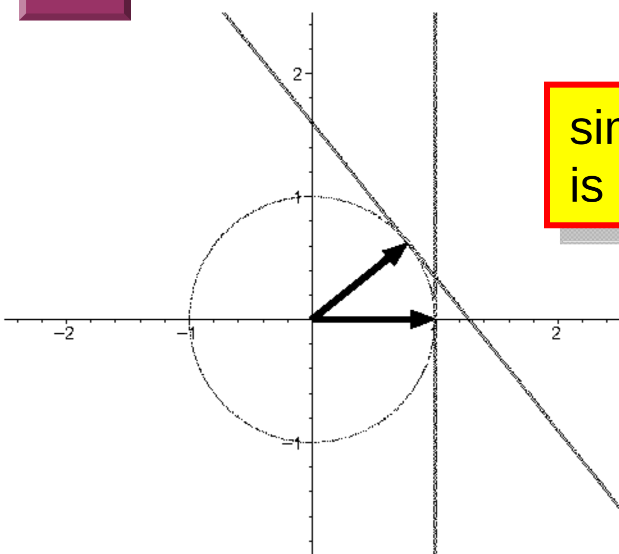
Euclid's
Geometry
(300 BC)

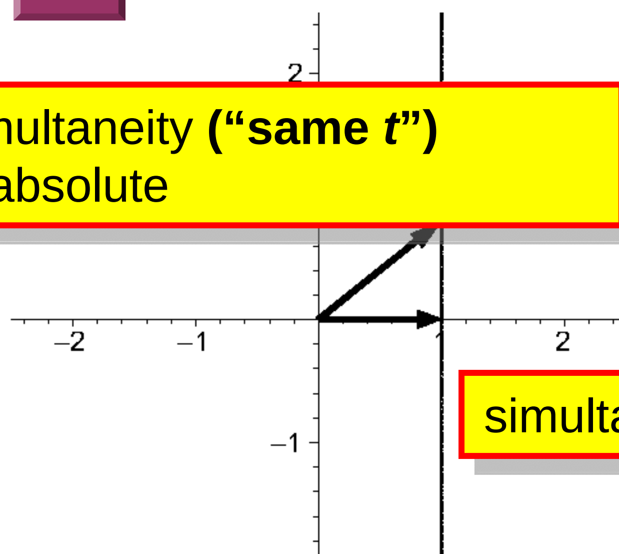


Galileo's
Relativity
(1632)

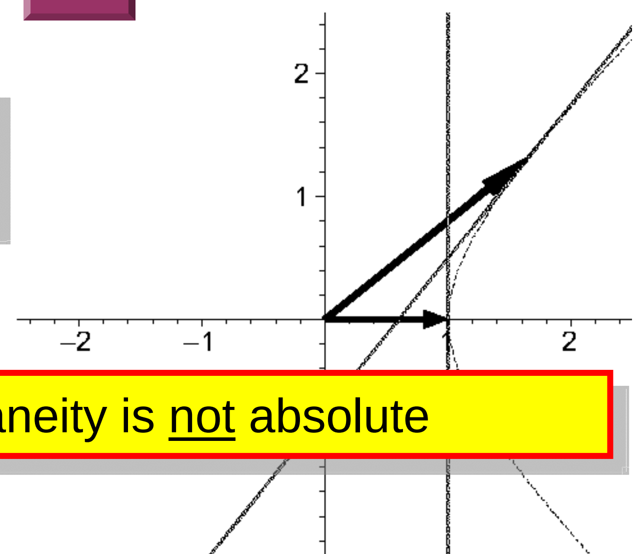
Einstein's
Relativity
(1905)

Minkowski's
Spacetime
Geometry
(1908)





simultaneity **("same *t*")**
is absolute



simultaneity is <u>not</u> absolute

inspired by the "Parable of the Surveyors" in
<u>Spacetime Physics</u> by Taylor and Wheeler

CIRCLES and the METRIC (separation of points)

Proper [Wristwatch] time, "Space"

$$R^2 = t^2 - \epsilon^2\ y^2$$

$$\begin{aligned} ds^2 &= (d\vec{s})^\top \widetilde{G}\,(d\vec{s}) \\ &= \begin{pmatrix} dt & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\boldsymbol{\epsilon}^2 \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix} \\ &= dt^2 - \boldsymbol{\epsilon}^2\ dy^2 \end{aligned}$$

radius vector is a **timelike-vector** $ds^2 > 0$

spacelike-vector is tangent to the circle, perpendicular to **timelike**

null-vector has $ds^2 = 0$

spatial distance

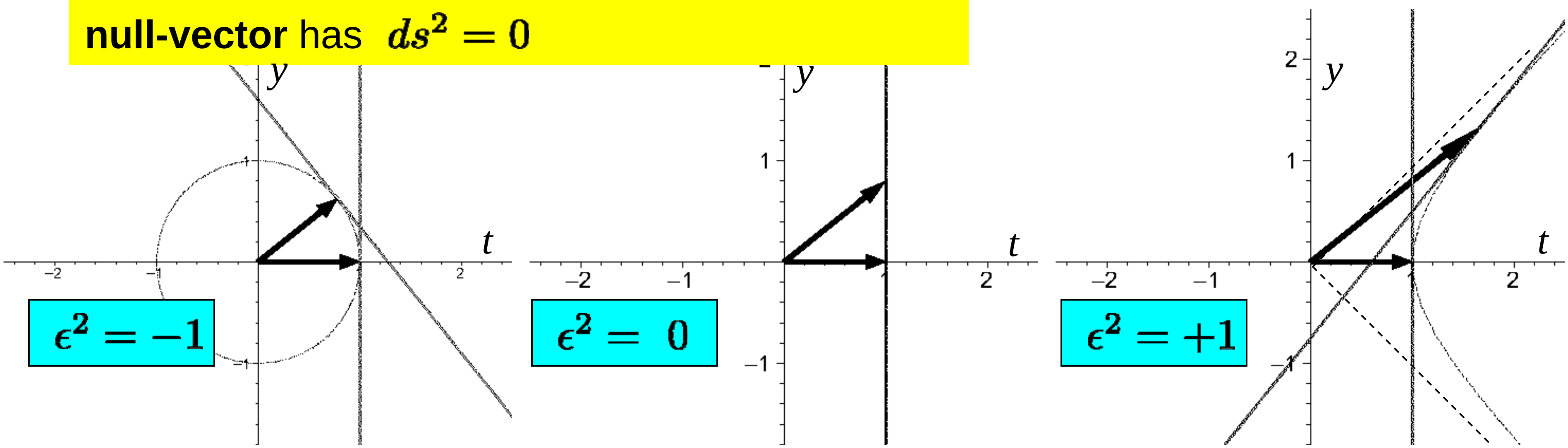
$\epsilon^2 \neq 0$

$$\begin{aligned} (dL)^2 &= -\frac{1}{\boldsymbol{\epsilon}^2}(dS)^2 \\ &= \begin{pmatrix} dt & dy \end{pmatrix} \begin{pmatrix} -\frac{1}{\boldsymbol{\epsilon}^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix} \\ &= (dy)^2 - \frac{1}{\boldsymbol{\epsilon}^2}(dt)^2 \end{aligned}$$

$\epsilon^2 = 0$

$$\begin{aligned} (dL)^2 &= (dy)^2 \\ &= \begin{pmatrix} dt & dy \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix} \end{aligned}$$

$$(dL)^2 = \begin{cases} (dy)^2 - \dfrac{1}{\boldsymbol{\epsilon}^2}(dt)^2 & \text{for } \boldsymbol{\epsilon}^2 \neq 0 \\ (dy)^2 & \text{for } \boldsymbol{\epsilon}^2 = 0 \end{cases}$$



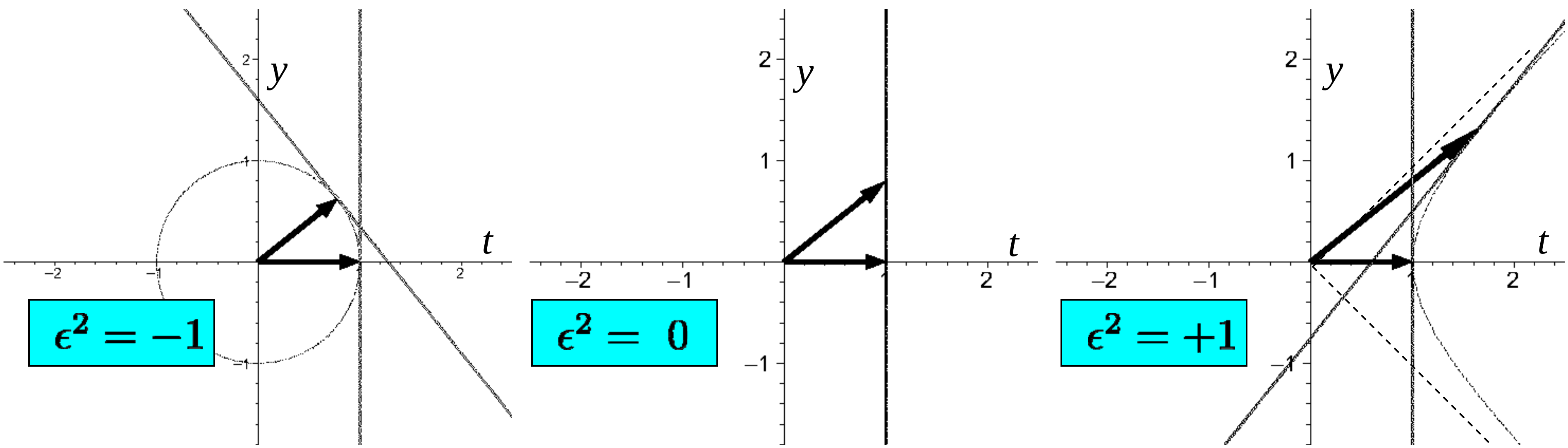
HYPERCOMPLEX NUMBERS

Maximum Signal Speed

$$R^2 = t^2 - \epsilon^2 \, y^2$$

Physically, $\epsilon^2 = (c_{light}/c_{max})^2$

Minkowskian	$c_{max} = c_{light}$ (finite):	$\epsilon^2 = 1$ [but $\epsilon \neq 1$] (double numbers)
Galilean	$c_{max} = \infty$ (infinite):	$\epsilon^2 = 0$ [but $\epsilon \neq 0$] (dual numbers)
Euclidean	$c_{max} = i c_{light}$ (finite, imaginary):	$\epsilon^2 = -1$ (complex numbers)



HYPERCOMPLEX NUMBERS

Maximum Signal Speed

$$R^2 = t^2 - \boxed{\epsilon^2}\, y^2$$

It is convenient (but not necessary) to introduce the following "generalized complex" or "hypercomplex" number systems. Consider quantities of the form $\mathbf{z} = \mathbf{a} + \epsilon\mathbf{b}$, where a and b are real-numbers and ϵ is the "generalized imaginary number". These quantities can be given a matrix representation:

$$\tilde{\epsilon} = \begin{pmatrix} 0 & \epsilon^2 \\ 1 & 0 \end{pmatrix}$$

real	$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	
complex	$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	**Euclidean**
dual ($\epsilon^2 = 0$ $[\epsilon \neq 0]$)	$\epsilon = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$	**Galilean**
double ($\epsilon^2 = 1$ $[\epsilon \neq 1]$)	$\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$a + \epsilon b = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$	**Minkowskian**

(these number systems have "divisors of zero")

Dual ($\epsilon^2 = 0$ $[\epsilon \neq 0]$):

$1/\epsilon$ implies
$$\begin{aligned} \epsilon x &= 1 \\ \epsilon^2 x &= \epsilon \\ 0 &= \epsilon \end{aligned}$$
impossible!

Double ($\epsilon^2 = 1$ $[\epsilon \neq 1]$):

$1/(1+\epsilon)$ implies
$$\begin{aligned} (1+\epsilon)x &= 1 \\ (1-\epsilon^2)x &= 1-\epsilon \\ 0 &= 1-\epsilon \end{aligned}$$
impossible!

Do formal calculations in which ϵ is treated algebraically but never evaluated until the last step.

All physical quantities involve ϵ^2 alone.

ANGLE (separation of lines)

Rapidity

$$\Theta = \frac{1}{R}\int dL$$

$$(dL)^2 = \begin{cases} (dy)^2 - \frac{1}{\epsilon^2}(dt)^2 & \text{for } \epsilon^2 \neq 0 \\ (dy)^2 & \text{for } \epsilon^2 = 0 \end{cases}$$

$$\Theta = \frac{1}{R}\int \sqrt{dy^2 - \frac{1}{\epsilon^2}dt^2} = \int \frac{dy}{\sqrt{R^2+\epsilon^2 y^2}} = \frac{1}{\epsilon}\sinh^{-1}(\epsilon y/R)$$

$$\epsilon y = R\sinh(\epsilon\Theta)$$
$$t = R\cosh(\epsilon\Theta).$$

Euclidean case ($\epsilon^2 = -1$)

$$y = R\sin(\theta_e)$$
$$t = R\cos(\theta_e)$$

Minkowskian case ($\epsilon^2 = +1$)

$$y = R\sinh(\theta_m)$$
$$t = R\cosh(\theta_m)$$

case ($\epsilon^2 = 0$),

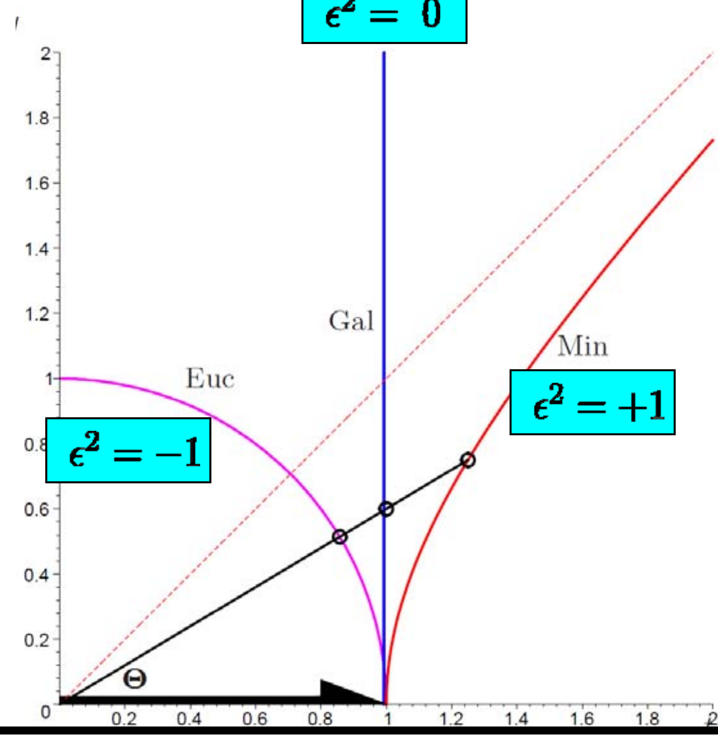
$$\theta_g = \frac{1}{R}\int dL = \frac{1}{R}\int dy = \frac{y}{R}$$

$$y = R\theta_g = R\ \mathbf{sing}(\theta_g)$$
$$t = R = R\ \mathbf{cosg}(\theta_g)$$

$$\mathbf{cosg}(\theta_g) = 1 \text{ and } \mathbf{sing}(\theta_g) = \theta_g$$

(Yaglom) **Galilean Trig Functions**

GENERALIZED Trig Functions

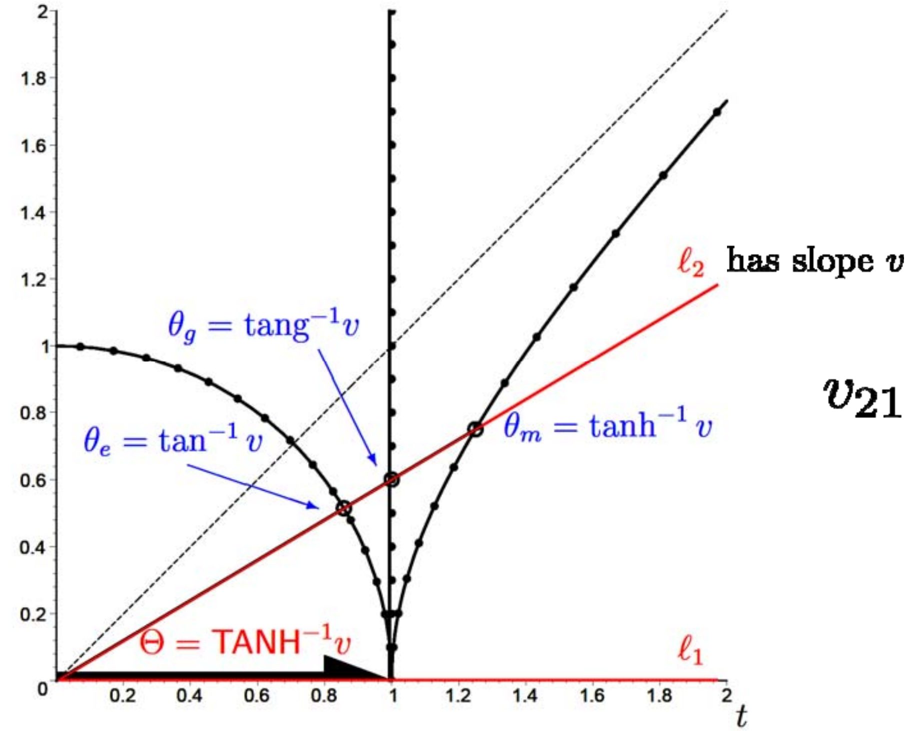


$$y = R\ \text{SINH}\ \Theta$$
$$t = R\ \text{COSH}\ \Theta$$

SLOPE = TANGENT(ANGLE)

Velocity = TANH (Rapidity)

$$y = R \ \mathrm{SINH}\,\Theta$$
$$t = R \ \mathrm{COSH}\,\Theta$$



$$\begin{aligned} v &= \frac{\Delta y}{\Delta t} \\ &= \mathrm{TANH}\,\Theta \\ &= \tan\theta_e = \mathrm{tang}\,\theta_g = \tanh\theta_m \end{aligned}$$

$$\begin{aligned} v_{21} &= \mathrm{TANH}(\Theta_{21}) \\ &= \mathrm{TANH}(\Theta_{20} - \Theta_{10}) \\ &= \frac{\mathrm{TANH}\,\Theta_{20} - \mathrm{TANH}\,\Theta_{10}}{1 - \epsilon^2\,\mathrm{TANH}\,\Theta_{20}\,\mathrm{TANH}\,\Theta_{10}} \end{aligned}$$

EUC	$\epsilon^2 = -1$	v_{21}	$=$	$\dfrac{v_{20} - v_{10}}{1 + v_{20}v_{10}}$
GAL	$\epsilon^2 = \ 0$	v_{21}	$=$	$v_{20} - v_{10}$
MIN	$\epsilon^2 = +1$	v_{21}	$=$	$\dfrac{v_{20} - v_{10}}{1 - v_{20}v_{10}}$

EULER and TRIGONOMETRIC functions

Relativistic "factors"

$$
\begin{aligned}
\text{EXP }\Theta &\equiv \exp(\epsilon\Theta)\\
&= 1+(\epsilon\Theta)+\frac{(\epsilon\Theta)^2}{2!}+\frac{(\epsilon\Theta)^3}{3!}+\frac{(\epsilon\Theta)^4}{4!}+\cdots\\
&= \left[1+\frac{(\epsilon\Theta)^2}{2!}+\frac{(\epsilon\Theta)^4}{4!}+\cdots\right]\\
&\qquad +\left[(\epsilon\Theta)+\frac{(\epsilon\Theta)^3}{3!}+\frac{(\epsilon\Theta)^5}{5!}+\cdots\right]\\
&= \cosh\,(\epsilon\Theta)+\sinh\,(\epsilon\Theta)\\
&= \left[1+\epsilon^2\frac{\Theta^2}{2!}+\epsilon^4\frac{\Theta^4}{4!}+\cdots\right]\\
&\qquad +\epsilon\left[\Theta+\epsilon^2\frac{\Theta^3}{3!}+\epsilon^4\frac{\Theta^5}{5!}+\cdots\right]\\
&= \text{COSH }\Theta+\epsilon\text{ SINH }\Theta
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \tanh\theta_m\\
\gamma &= \cosh\theta_m\\
\beta\gamma &= \sinh\theta_m\\
k &= \cosh\theta_m+\sinh\theta_m\\
&= \exp\theta_m
\end{aligned}
$$

$$\epsilon\text{ TANH }\Theta \equiv \tanh(\epsilon\Theta)=\frac{\sinh(\epsilon\Theta)}{\cosh(\epsilon\Theta)}=\frac{\epsilon\text{ SINH }\Theta}{\text{COSH }\Theta}$$

$$\text{EXP}(-\Theta)=\text{COSH }\Theta-\epsilon\text{ SINH }\Theta$$

$$\text{EXP}(-\Theta)=\frac{1}{\text{EXP}(\Theta)}$$

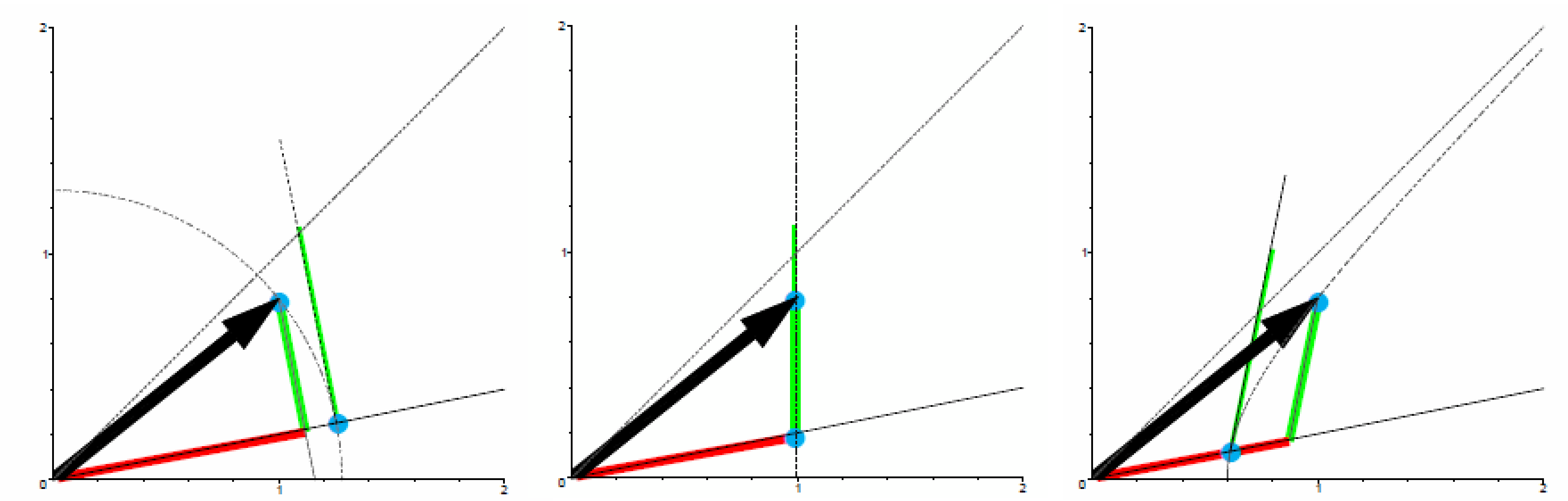
Algebraic Identities

$$
\begin{aligned}
1 &= \text{COSH}^2\Theta-\epsilon^2\text{SINH}^2\Theta\\
\text{TANH }(\Theta_1+\Theta_2) &= \frac{\text{TANH }\Theta_1+\text{TANH }\Theta_2}{1+\epsilon^2\text{ TANH }\Theta_1\text{ TANH }\Theta_2}\\
\text{COSH }\Theta &= (1-\epsilon^2\text{ TANH}^2\Theta)^{-1/2}
\end{aligned}
$$

Differential Identities

$$
\begin{aligned}
\frac{d}{d\Theta}\text{EXP }\Theta &= \epsilon\text{ EXP }\Theta\\
\frac{d}{d\Theta}\text{COSH }\Theta &= \epsilon^2\text{ SINH }\Theta\\
\frac{d}{d\Theta}\text{SINH }\Theta &= \text{COSH }\Theta\\
\frac{d}{d\Theta}\text{TANH }\Theta &= 1-\epsilon^2\text{ TANH }^2\Theta
\end{aligned}
$$

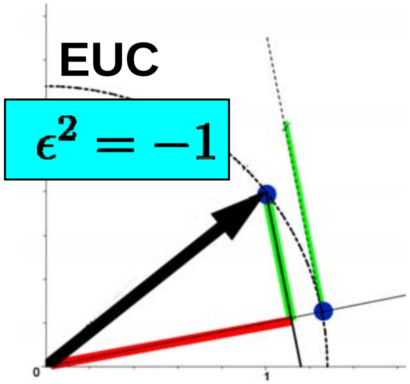
Every vector can be thought of as the HYPOTENUSE of some RIGHT triangle.

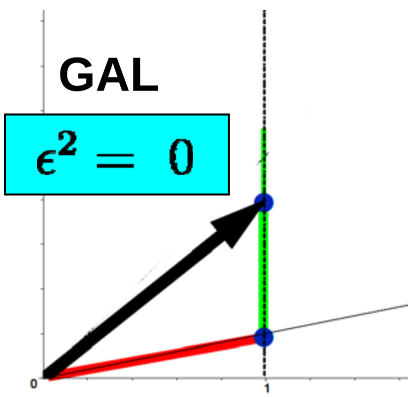


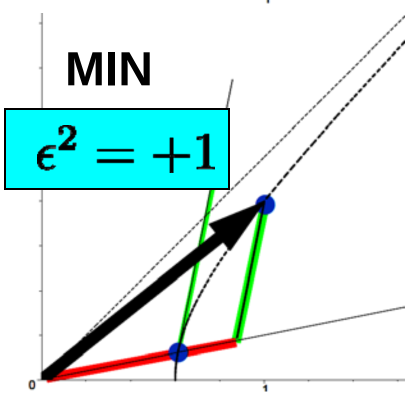
Project the vector into components parallel and perpendicular to a given direction.
"Drop the perpendicular" by constructing parallels to the tangent of the circle.

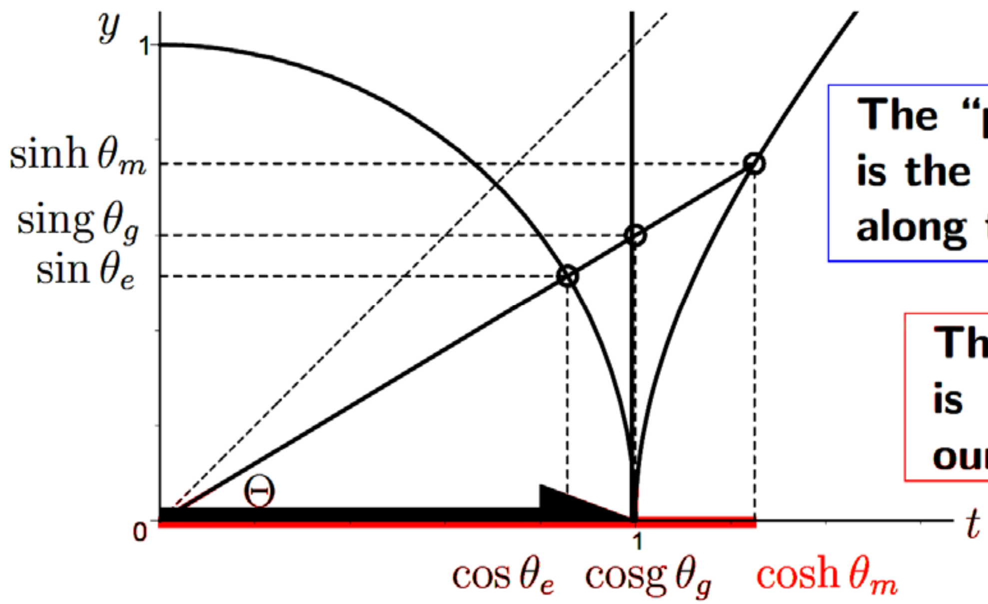
Projection onto a line

Time dilation



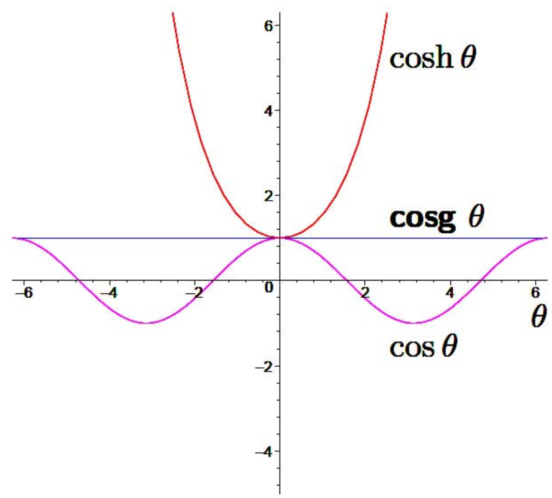






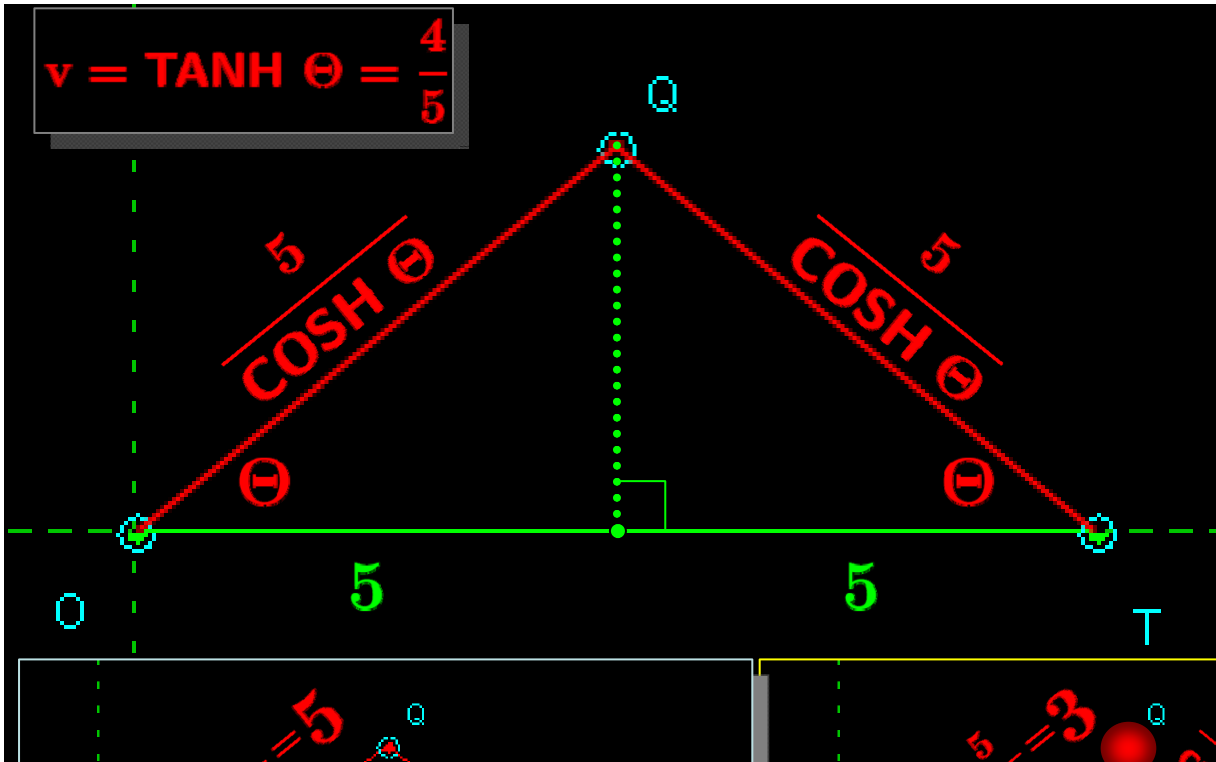
The "proper time" is the hypotenuse along the radius.

The "apparent time" is the projection onto our observer's t-axis.



an example of applied Spacetime Trigononometry

The Clock Effect / Twin Paradox



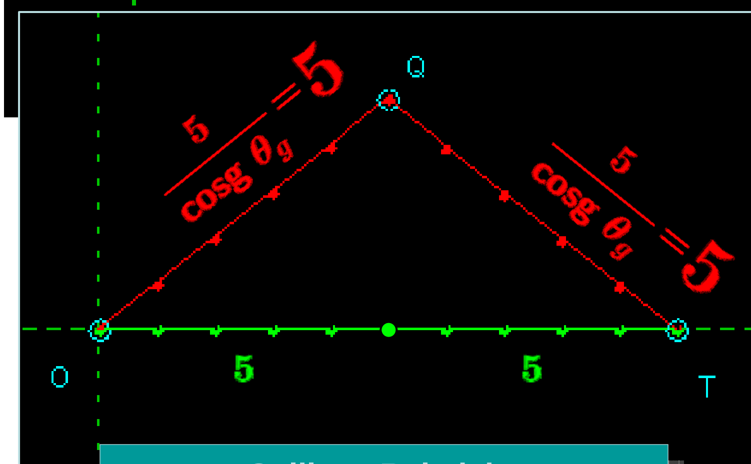
$$\text{COSH}^2\Theta - \epsilon^2\text{SINH}^2\Theta = 1$$

$$\text{COSH}^2\Theta\left(1 - \epsilon^2\frac{\text{SINH}^2\Theta}{\text{COSH}^2\Theta}\right) = 1$$

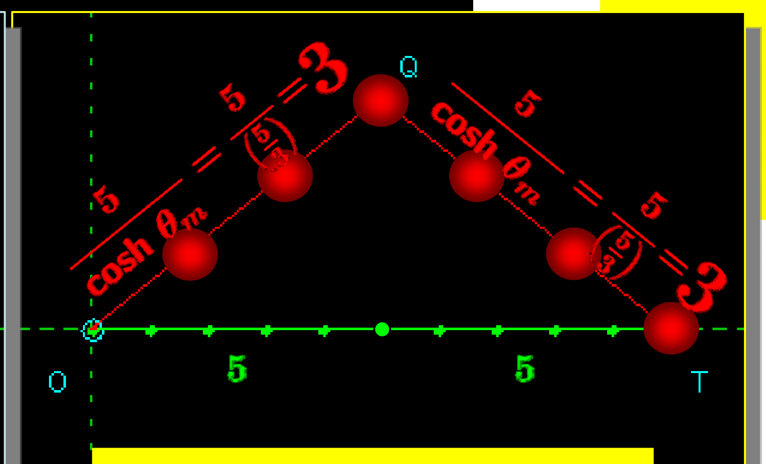
$$\text{COSH}\ \Theta = \frac{1}{\sqrt{1 - \epsilon^2\text{TANH}^2\Theta}}$$

$$\epsilon^2 = \begin{array}{rl} -1 & \text{Euclidean} \\ 0 & \text{Galilean} \\ +1 & \text{Minkowskian} \end{array}$$

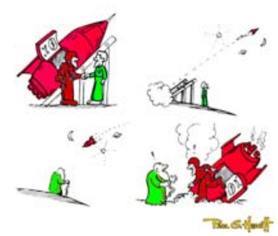
$$\cosh\ \theta_m = \frac{1}{\sqrt{1-(+1)\tanh^2\theta_m}} = \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \frac{5}{\sqrt{5^2-4^2}} = \frac{5}{3}$$



Galilean Relativity

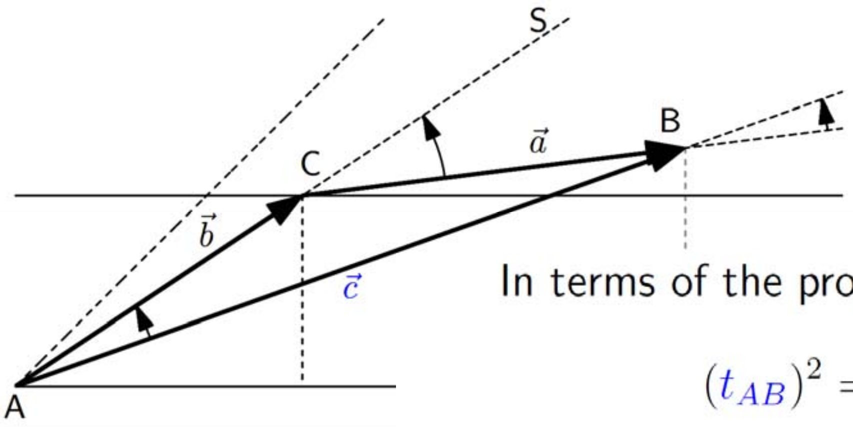


Special Relativity



Law of COSINES

Clock Effect

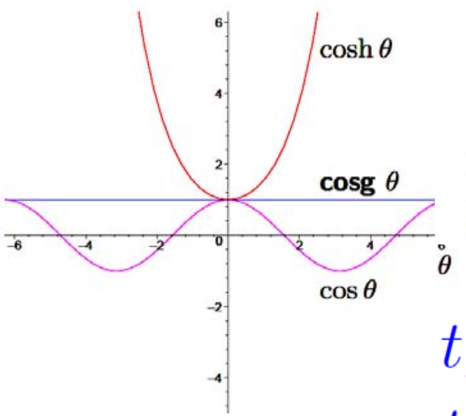


$$\begin{aligned} \vec{c} &= \vec{b} + \vec{a} \\ \vec{c}\cdot\vec{c} &= \vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a} + 2\vec{b}\cdot\vec{a} \\ c^2 &= b^2 + a^2 + 2ba \text{ COSH } (m\angle BCS) \end{aligned}$$

In terms of the proper-time elapsed,

$$(t_{AB})^2 = \boxed{(t_{AC})^2 + (t_{CB})^2} + 2t_{AC}t_{CB} \text{ COSH } (m\angle BCS)$$

Comparing this with the identity



$$(t_{AC} + t_{CB})^2 = \boxed{(t_{AC})^2 + (t_{CB})^2} + 2t_{AC}t_{CB}$$

and using the facts that $\cos\theta_e \le 1$, $\operatorname{cosg}\theta_g = 1$, and $\cosh\theta_m \ge 1$, the Law of Cosines implies the following relations:

$t_{AB} < t_{AC} + t_{CB}$ for $\epsilon^2 = -1$ "triangle inequality"

$t_{AB} = t_{AC} + t_{CB}$ for $\epsilon^2 = 0$ non-"clock effect"

$t_{AB} > t_{AC} + t_{CB}$ for $\epsilon^2 = +1$ "clock effect"

Rotations

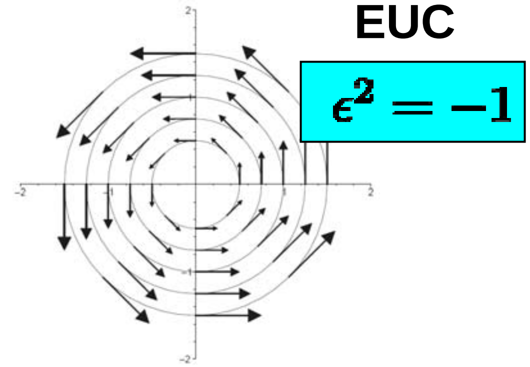
Boost transformations

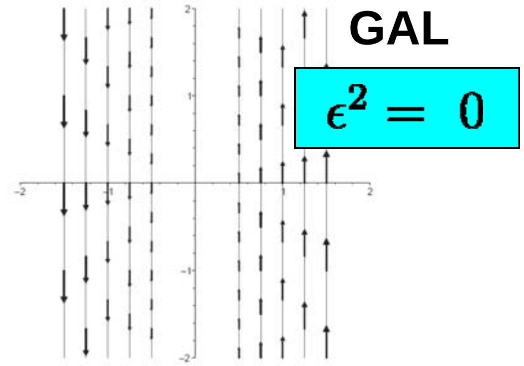
Consider a linear transformation $\vec{V}' = R(\Theta)\vec{V}$, where R satisfies:

$$\det R = 1 \qquad R(0) = I$$
$$R^T G R = G \qquad R(\Theta)R(\Phi) = R(\Theta + \Phi)$$

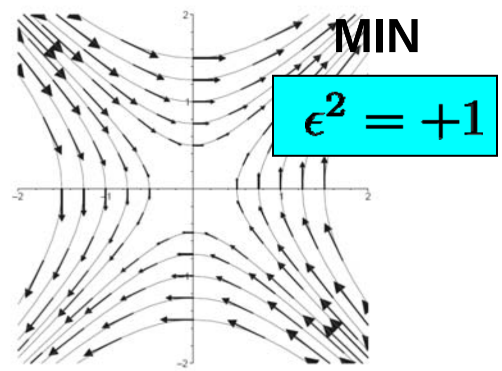
In terms of an orthogonal basis $\{\hat{t}, \hat{y}\}$ with metric $G = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon^2 \end{pmatrix}$

$$R(\Theta) = \begin{pmatrix} \text{COSH } \Theta & \epsilon^2 \text{SINH } \Theta \\ \text{SINH } \Theta & \text{COSH } \Theta \end{pmatrix}$$ is a "rotation" for that metric.





$$\begin{pmatrix} \operatorname{cosg}\theta_g & 0 \\ \operatorname{sing}\theta_g & \operatorname{cosg}\theta_g \end{pmatrix} = \operatorname{cosg}\theta_g \begin{pmatrix} 1 & 0 \\ \operatorname{tang}\theta_g & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$



$$\begin{pmatrix} \cosh\theta_m & \sinh\theta_m \\ \sinh\theta_m & \cosh\theta_m \end{pmatrix} = \cosh\theta_m \begin{pmatrix} 1 & \tanh\theta_m \\ \tanh\theta_m & 1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$$

Eigenvectors and Eigenvalues

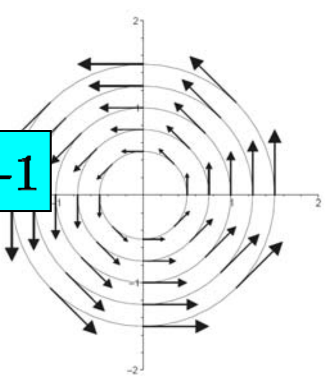
"Absolute" invariants

$$R(\Theta) = \begin{pmatrix} \text{COSH } \Theta & \epsilon^2 \text{SINH } \Theta \\ \text{SINH } \Theta & \text{COSH } \Theta \end{pmatrix}$$

	eigenvalue	eigenvectors
EUC		$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (better: invariant vector)
GAL	1 "absolute length"	$\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "absolute time"
MIN	$\cosh\theta \pm \sinh\theta = \exp(\pm\theta)$ $= \sqrt{\dfrac{1 \pm v}{1 \mp v}}$ Doppler-Bondi factor	$\hat{k} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\hat{l} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ "absolute speed of light"

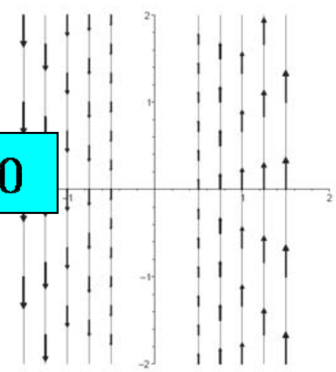
EUC

$\epsilon^2 = -1$



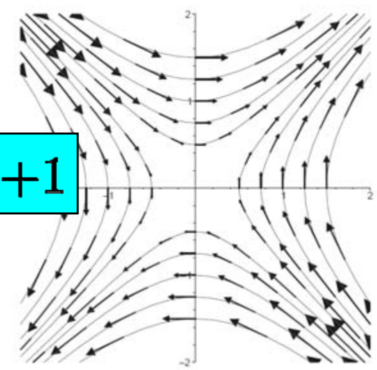
GAL

$\epsilon^2 = 0$



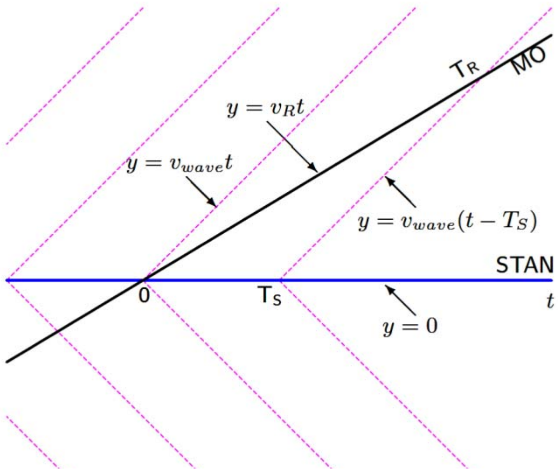
MIN

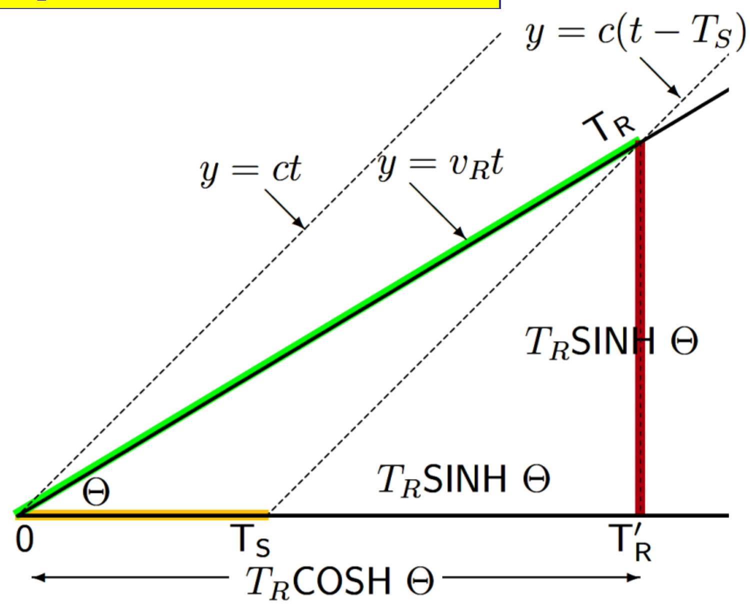
$\epsilon^2 = +1$



An interesting trigonometry problem

Doppler effect (unified)





Moving [Receding] Receiver

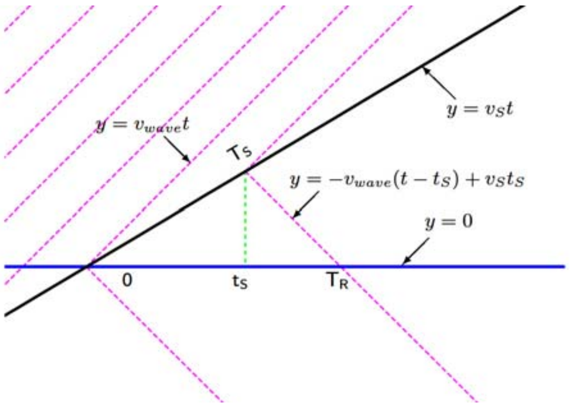
$$
\begin{aligned}
T_S &= T_R(\text{COSH }\Theta - \text{SINH }\Theta) \\
&= T_R\ \textbf{COSH}\,\Theta(1 - \textbf{TANH}\,\Theta) \\
&= T_R(1)(1 - v)\ \textbf{Gal} \\
&= T_R \frac{1}{\sqrt{1 - v^2}}(1 - v)\ \textbf{Min} \\
&= T_R\sqrt{\frac{1 - v}{1 + v}}
\end{aligned}
$$

$$f_o = \left(1 - \frac{v_o}{v}\right) f_s$$ *(observer moving away from a stationary source)*

$$
\nu_R = \begin{cases} \nu_S\left(1 - \frac{v}{c}\right) & \text{Gal} \\ \nu_S\sqrt{\dfrac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} & \text{Min} \end{cases}
$$

An interesting trigonometry problem

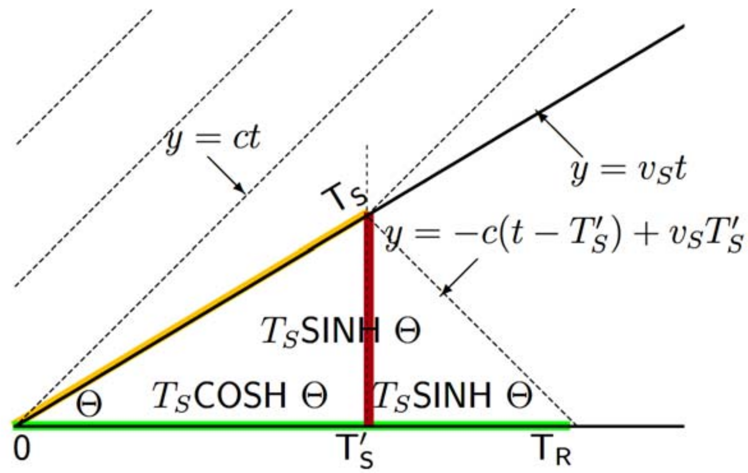
Doppler effect (unified)



Moving [Receding] Source

$$T_R = T_S(\text{COSH } \Theta + \text{SINH } \Theta)$$

$$T_S = T_R \frac{1}{(\text{COSH } \Theta + \text{SINH } \Theta)}$$



$$f_o = \left(\frac{v}{v + v_s}\right) f_s = \left(\frac{1}{1 + \frac{v_s}{v}}\right) f_s \quad \text{(source moving away from a stationary observer)} \qquad (14.10)$$

$$\nu_R = \begin{cases} \nu_S \dfrac{1}{(1 + \frac{v}{c})} & \text{Gal} \\ \nu_S \sqrt{\dfrac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} & \text{Min} \end{cases}$$

Moving [Receding] Receiver

$$T_S = T_R(\text{COSH } \Theta - \text{SINH } \Theta)$$

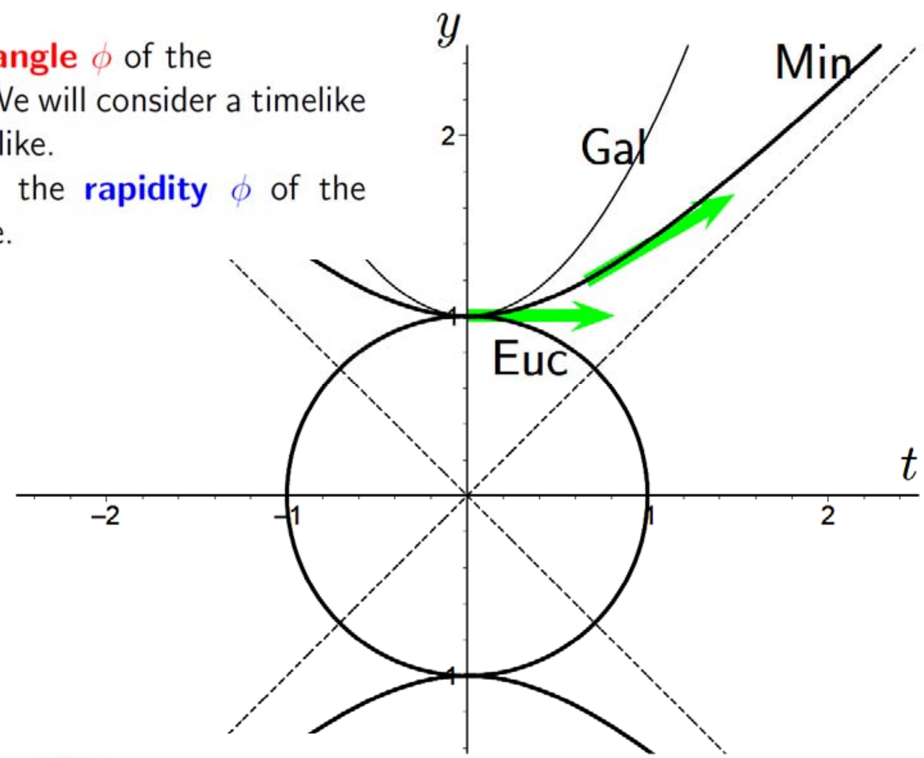
$$\nu_R = \begin{cases} \nu_S(1 - \frac{v}{c}) & \text{Gal} \\ \nu_S \sqrt{\dfrac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} & \text{Min} \end{cases}$$

Curve of constant curvature

Uniformly accelerated observer (unified)

The **curvature** ρ of a plane curve is a measure of how the **angle** ϕ of the **tangent vector** $\dot{y}$ changes with **arc-length** s along the curve. We will consider a timelike plane curve $y(t)$, i.e., a curve whose tangent is everywhere timelike.

The **acceleration** ρ of a worldline is a measure of how the **rapidity** ϕ of the **velocity vector** $\dot{y}$ changes with **proper-time** s along the curve.



$$\rho \equiv \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = \frac{\ddot{y}}{[1-\epsilon^2(\dot{y})^2]^{3/2}}$$

We seek the **curve of constant curvature:** $\rho = a_0$:

If $\epsilon^2 \neq 0$,

$$(y-y_0)^2 - \frac{1}{\epsilon^2}(t-t_0)^2 = \frac{1}{\epsilon^4}{a_0}^{-2} \qquad \begin{cases} \text{if } \epsilon^2 = -1 & \textbf{circle} \\ \text{if } \epsilon^2 = +1 & \textbf{hyperbola} \end{cases}$$

If $\epsilon^2 = 0$,

$$y - y_0 = \frac{1}{2}a_0(t-t_0)^2 \qquad \{ \quad \textbf{parabola}$$

EUCLID's FIRST

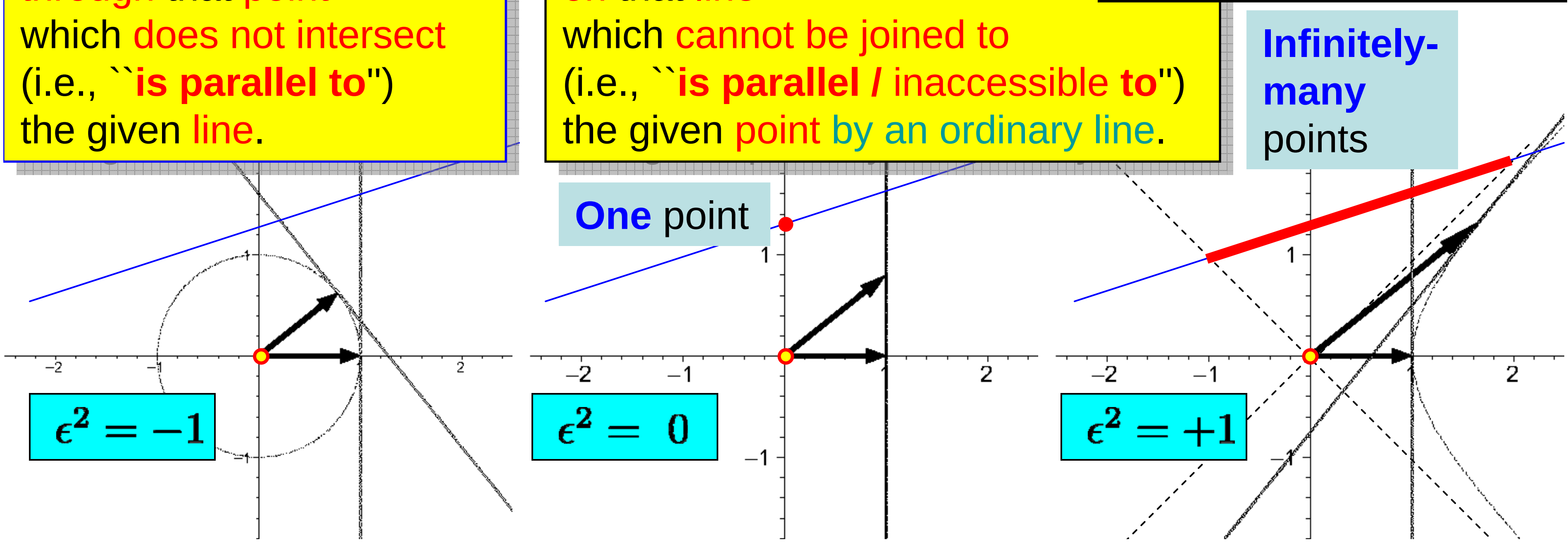
Causal Structure of Spacetime

EUCLID's FIRST
"To draw a straight line from any point to any point."

EUCLID's FIFTH
(Playfair)
Given a line and
a point not on that line,
there exists
precisely <u>one</u> line
through that point
which does not intersect
(i.e., ``**is parallel to**'')
the given line.

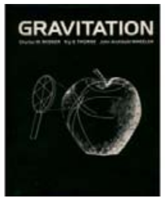
EUCLID's FIRST
(like Playfair dualized)
Given a point and
a line not through that point,
there exists
<u>no</u> point
on that line
which cannot be joined to
(i.e., ``**is parallel /** inaccessible **to**'')
the given point by an ordinary line.

Spacetime geometries fail Euclid's First Postulate!



advanced topic: **Visualizing Tensor Algebra**

The "circle" is a visualization of its metric tensor $\mathbf{g}_{ab}$!



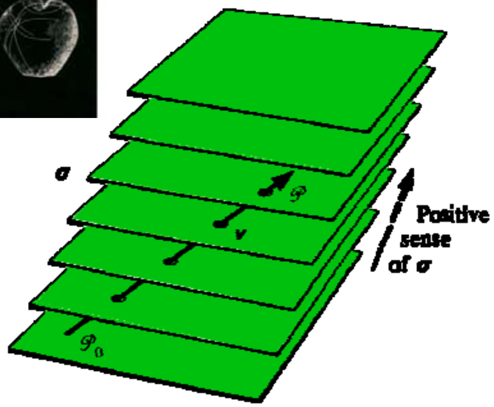
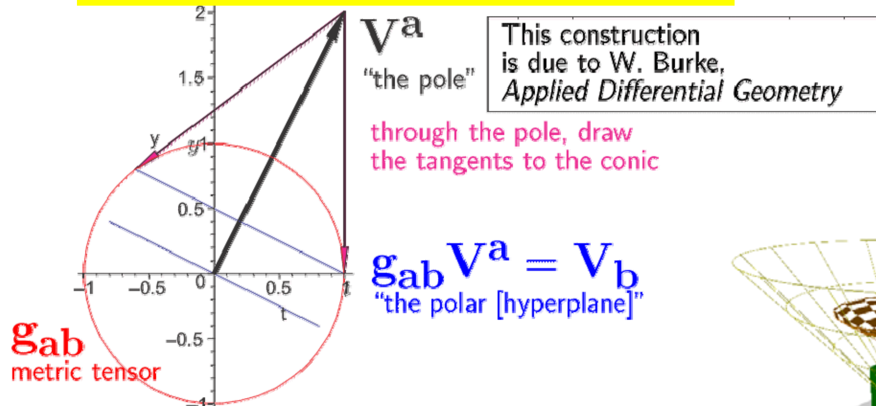
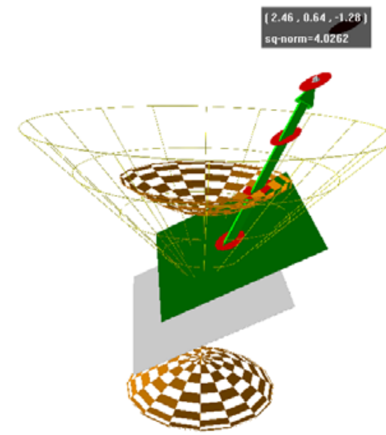


Figure 2.4.
The vector separation $v = \mathscr{P} - \mathscr{P}_0$ between two neighboring events $\mathscr{P}_0$ and $\mathscr{P}$; a 1-form σ; and the piercing of σ by v to give the number

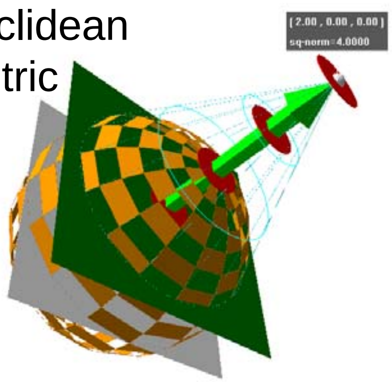
$$\langle \sigma, v \rangle = (\text{number of surfaces pierced}) = 4.4$$

(4.4 "bongs of bell"). When σ is made of surfaces of constant phase,

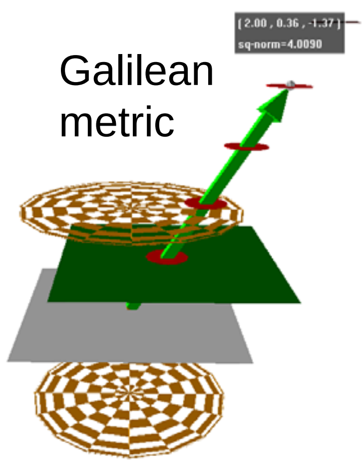


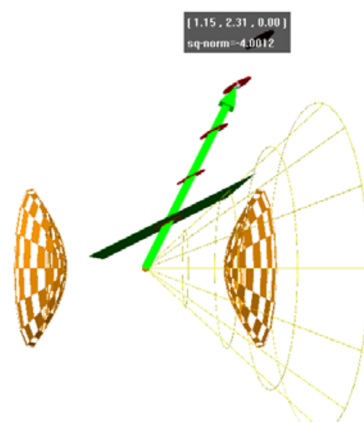


Euclidean metric

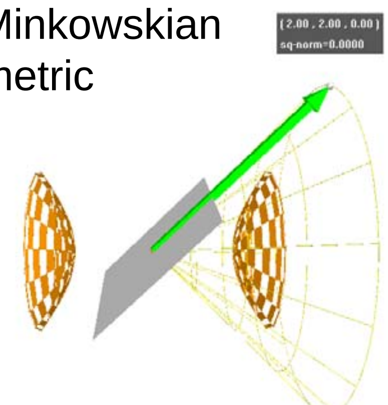


Galilean metric





Minkowskian metric



Some problems I am working on:

Interpret the “Law of Sines” physically.
(Interpret a result from [Euclidean] geometry in terms of a physical situation in spacetime.)

Collisions (in Galilean and Special Relativity) –
elastic collisions, inelastic collisions, coefficient of restitution;
energy (Kinetic and Rest energy), spatial-momentum

Hypercomplex numbers – do Geometry as one does with Complex Numbers
(Dual numbers are used in robotics. How?)

Differential Geometry with “degenerate metrics” (Galilean limits)

Connection to Norman Wildberger’s Universal Hyperbolic Trigonometry?

Electromagnetism (Maxwell’s Equations)
Galilean-invariant version (Jammer and Stachel)
“If Maxwell had worked between Ampere and Faraday?”

De Sitter spacetimes as analogues of Elliptic and Hyperbolic Geometries

conclusions

- Cayley-Klein geometry
  provides **geometrical analogies**
  which can be given **kinematical interpretations**
  …. starting with the Galilean case,
  onto the Special Relativistic case,
  and further onto the deSitter spacetimes
  (the simplest General Relativistic cases)

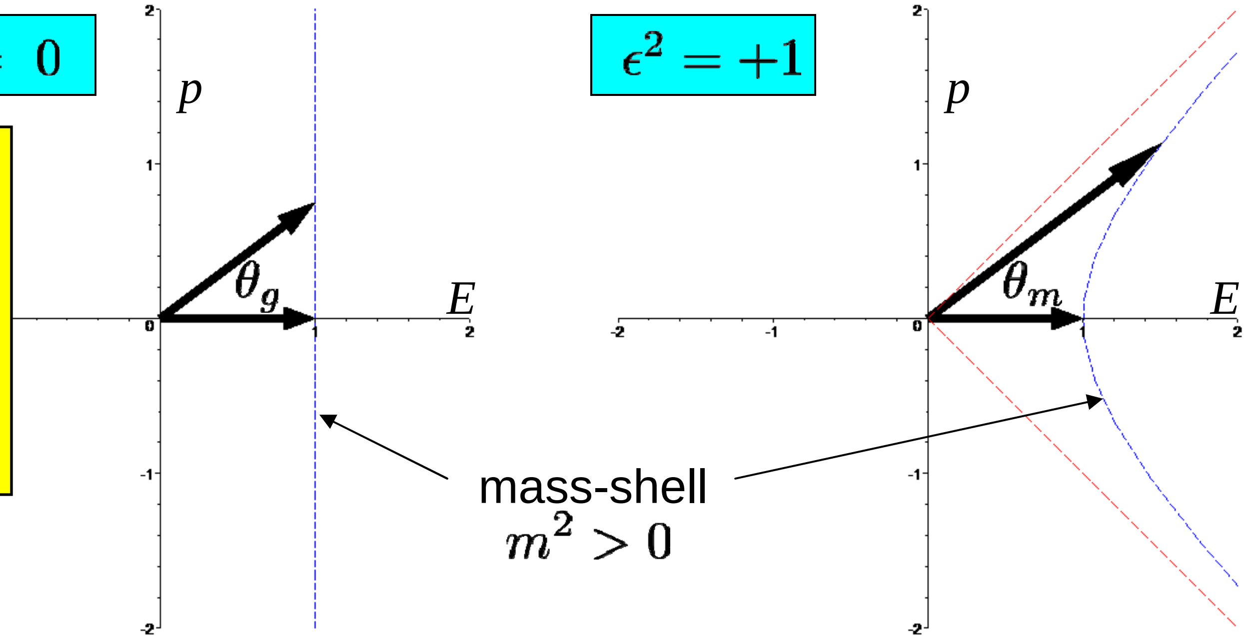
- may be an **easier approach to learning Relativity**

- **Galilean Limits are clarified**

Energy-Momentum Space

$\epsilon^2 = 0$ $\epsilon^2 = +1$

Two identical particles
with mass ***m***:
one at rest
in this frame,
the other traveling with
velocity $v = \mathbf{TANH}(\Theta)$



In component-form, the energy-momentum vector is $\tilde{m} = \begin{pmatrix} m\,\mathbf{COSH}\ \Theta \\ m\,\mathbf{SINH}\ \Theta \end{pmatrix}$

$\epsilon^2 = 0$ $$\tilde{m} = \begin{pmatrix} m\,\mathbf{cosg}\ \theta_g \\ m\,\mathbf{sing}\ \theta_g \end{pmatrix} = \begin{pmatrix} m(\ 1\ ) \\ m(\ \theta_g\ ) \end{pmatrix} = \begin{pmatrix} m \\ mv \end{pmatrix}$$

Note: In the Galilean case, the "energy component" is always the "rest mass".

$\epsilon^2 = +1$ $$\tilde{m} = \begin{pmatrix} m\cosh\ \theta_m \\ m\sinh\ \theta_m \end{pmatrix} = \begin{pmatrix} m\cosh\ \theta_m \\ m\cosh\ \theta_m \tanh\ \theta_m \end{pmatrix} = \begin{pmatrix} \gamma m \\ \gamma m v \end{pmatrix}$$

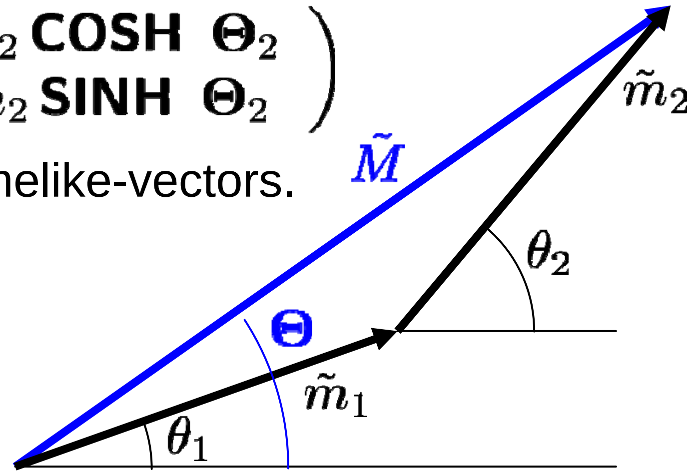
Conservation of Energy-Momentum

A particle with rest-mass M decays into two particles with rest-masses m_1 and m_2.

Conservation: $\tilde{M} = \tilde{m}_1 + \tilde{m}_2$

$$\begin{pmatrix} M \cosh \Theta \\ M \sinh \Theta \end{pmatrix} = \begin{pmatrix} m_1 \cosh \Theta_1 \\ m_1 \sinh \Theta_1 \end{pmatrix} + \begin{pmatrix} m_2 \cosh \Theta_2 \\ m_2 \sinh \Theta_2 \end{pmatrix}$$

Geometrically, this is a triangle formed with future-timelike-vectors.



$\epsilon^2 = 0$

$$\begin{pmatrix} M \\ MV \end{pmatrix} = \begin{pmatrix} m_1 \\ m_1 v_1 \end{pmatrix} + \begin{pmatrix} m_2 \\ m_2 v_2 \end{pmatrix}$$

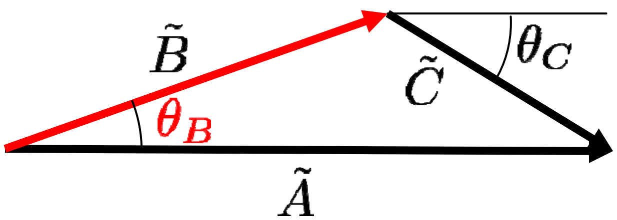
In the Galilean case, “energy conservation” implies “conservation of total rest mass”.

$\epsilon^2 = +1$

$$\begin{pmatrix} \Gamma M \\ \Gamma M V \end{pmatrix} = \begin{pmatrix} \gamma_1 m_1 \\ \gamma_1 m_1 v_1 \end{pmatrix} + \begin{pmatrix} \gamma_2 m_2 \\ \gamma_2 m_2 v_2 \end{pmatrix}$$

Energy-momentum
decay

Griffiths
(Elementary Particles)



3.16. Particle A, at rest, decays into particles B and C ($A \to B + C$).

(a) Find the energy of the outgoing particles, in terms of the various masses.

$$\left[\textit{Answer:} \qquad E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}c^2\right]$$

(b) Find the magnitudes of the outgoing momenta.

$$\left[\textit{Answer:} \qquad |\mathbf{p}_B| = |\mathbf{p}_C| = \frac{\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}}{2m_A}c,\right.$$

where λ is the so-called *triangle function:*

$$\left.\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.\right]$$

(c) Note that λ factors: $\lambda(a^2, b^2, c^2) = (a + b + c)(a + b - c)(a - b + c)(a - b - c)$. Thus $|\mathbf{p}_B|$ goes to zero when $m_A = m_B + m_C$, and runs imaginary if $m_A < (m_B + m_C)$. Explain.

- Law of Cosines: $C^2 = A^2 + B^2 - 2\tilde{A}\cdot\tilde{B}$

$$m_C^2 = m_A^2 + m_B^2 - 2m_A m_B \,\mathbf{COSH}\,\Theta_B$$

so $\quad E_B = \hat{A}\cdot\tilde{B} = m_B\,\mathbf{COSH}\,\Theta_B = (m_A^2 + m_B^2 - m_C^2)/(2m_A)$

- Law of Sines yields: $\dfrac{\mathbf{SINH}\,\Theta_C}{m_B} = \dfrac{\mathbf{SINH}\,\Theta_B}{m_C}$

Generalized Heron formula

Multiply by half of the product of the three masses

$$\frac{m_A m_C\,\mathbf{SINH}\,\Theta_C}{2} = \frac{m_A m_B\,\mathbf{SINH}\,\Theta_B}{2} = \left(\begin{array}{c}\text{triangle}\\ \text{area}\end{array}\right) = \frac{\sqrt{\lambda}}{4}$$

$$p_B = \|\hat{A}\times\tilde{B}\| = \frac{m_A m_B\,\mathbf{SINH}\,\Theta_B}{m_A} = \frac{\sqrt{\lambda}}{2m_A}$$

Galilean-invariant Electromagnetism

(Jammer and Stachel)

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\alpha \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} = \vec{j} + \beta \frac{\partial \vec{D}}{\partial t}$$

with $\vec{D} = \kappa \vec{E}$ **and** $\vec{B} = \mu \vec{H}$**.**